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Oscillation criteria for half-linear elliptic inequalities with $p(x)$-Laplacians via Riccati method ☆

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ABSTRACT

Riccati inequality for half-linear elliptic inequalities with $p(x)$-Laplacians is established, and oscillation criteria are derived by using the Riccati inequality. Our method is to reduce oscillation problems for half-linear elliptic inequalities with $p(x)$-Laplacians to one-dimensional Riccati inequality with variable exponents. Generalizations to more general elliptic inequalities with $p(x)$-Laplacians are also discussed.

1. Introduction

Beginning with the work of Noussair and Swanson [1] dealing with semilinear elliptic inequalities, efforts have been made by numerous authors to generalize to more general elliptic equations, for example, quasilinear elliptic equations including half-linear elliptic equations.

In recent years there has been considerable investigation on oscillations of half-linear elliptic equations. In 1998 Usami [2] investigated oscillations of half-linear elliptic equations of the form

$$\nabla \cdot \left(a(x)|\nabla u|^{p-2}\nabla u\right) + c(x)|u|^{p-2}u = 0 \quad (p > 1)$$

by using Riccati techniques, where $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_n)$ and the dot $\cdot$ denotes the scalar product, and then oscillation results via Riccati method have been developed by several authors, see for example, [3–9], and the references cited therein.

The operator $-\nabla \cdot \left(|\nabla u|^{p(x)-2}\nabla u\right)$ is said to be $p(x)$-Laplacian, and becomes p-Laplacian $-\nabla \cdot \left(|\nabla u|^{p-2}\nabla u\right)$ if $p(x) = p$ (constant). Much current interest has been focused on various mathematical problems with variable exponent growth conditions (see [10]). The study of such problems arises from nonlinear elasticity theory, electrorheological fluids (cf. [11,12]). We mention, in particular, the paper [13] by Allegretto in which Picone identity arguments are used, and the formulae that are closely related to Picone identities and Riccati inequalities are established.

Existence of weak solutions of $p(x)$-Laplacian equations of the form

$$-\nabla \cdot \left(a(x)|\nabla u|^{p(x)-2}\nabla u\right) + |u|^{p(x)-2}u = f(x, u) \quad \text{in } \mathbb{R}^n$$

were investigated by several authors, see for example, [14–17].

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The paper [18] by Zhang seems to be the first paper dealing with oscillations of solutions of $p(x)$-Laplacian equations. In [18] oscillation problem for the $p(t)$-Laplacian equation

$$\left(|u'|^{p(t)-2}u'\right)' + t^{-\theta(t)}g(t,u) = 0, \quad t > 0$$

was treated. Motivated by Zhang [18], we study the oscillation properties of half-linear elliptic inequalities with $p(x)$-Laplacian via Riccati techniques.

We note that the following elliptic equation with $p(x)$-Laplacian ($p(x) = \alpha(x) + 1$)

$$\nabla \cdot \left(A(x)|\nabla v|^{\alpha(x)-1}\nabla v\right) + C(x)|v|^{\alpha(x)-1}v = 0$$

is not half-linear if $\alpha(x)$ is not a constant. However, it is checked that the elliptic inequality with $p(x)$-Laplacian ($p(x) = \alpha(x) + 1$)

$$vQ[v] \le 0 \tag{1.1}$$

is *half-linear* in the sense that a constant multiple of a solution v of (1.1) is also a solution of (1.1) (see Proposition 2.1 in Section 2), where

$$\begin{aligned} Q[v] &:= \nabla \cdot \left(A(x)|\nabla v|^{\alpha(x)-1}\nabla v\right) - A(x)(\log|v|)|\nabla v|^{\alpha(x)-1}\nabla\alpha(x)\cdot\nabla v \\ &\quad + |\nabla v|^{\alpha(x)-1}B(x)\cdot\nabla v + C(x)|v|^{\alpha(x)-1}v. \end{aligned} \tag{1.2}$$

There appears to be no known oscillation results for half-linear elliptic inequality (1.1). The objective of this paper is to obtain sufficient conditions for every solution v of (1.1) to be oscillatory in an exterior domain $\Omega \subset \mathbb{R}^n$, by utilizing Riccati inequality.

In Section 2 we establish Riccati inequality for (1.1), and in Section 3 we derive oscillation results for (1.1) on the basis of the Riccati inequality obtained in Section 2. Generalizations to more general elliptic inequalities are discussed in Section 4.

2. Riccati inequality

Let Ω be an exterior domain in $\mathbb{R}^n$, that is, Ω includes the domain $\{x \in \mathbb{R}^n;\ |x| \ge r_0\}$ for some $r_0 > 0$. It is assumed that $A(x) \in C(\Omega; (0,\infty))$, $B(x) \in C(\Omega; \mathbb{R}^n)$, $C(x) \in C(\Omega; \mathbb{R})$, and that $\alpha(x) \in C^1(\Omega; (0,\infty))$. The domain $\mathcal{D}_Q(\Omega)$ of Q is defined to be the set of all functions v of class $C^1(\Omega; \mathbb{R})$ such that $A(x)|\nabla v|^{\alpha(x)-1}\nabla v \in C^1(\Omega; \mathbb{R}^n)$.

We remark that $\log|v|$ in (1.2) has singularities at zeros of v, but $v\log|v|$ becomes continuous at the zeros of v if we define $v\log|v| = 0$ at the zeros, in view of the fact that $\lim_{\varepsilon\to+0}\varepsilon\log\varepsilon = 0$. Hence, we find that $vQ[v]$ has no singularities and is continuous in Ω.

Proposition 2.1. *Elliptic inequality* (1.1) *is half-linear in the sense that if v is a solution of* (1.1)*, then kv is also a solution of* (1.1) *for any constant k.*

Proof. Let v be any solution of (1.1), and $k(\ne 0)$ be any constant. We easily see that

$$\begin{aligned} Q[kv] &= \nabla \cdot \left(|k|^{\alpha(x)-1}kA(x)|\nabla v|^{\alpha(x)-1}\nabla v\right) - A(x)\left(|k|^{\alpha(x)-1}k\right)(\log(|k||v|))|\nabla v|^{\alpha(x)-1}\nabla\alpha(x)\cdot\nabla v \\ &\quad + \left(|k|^{\alpha(x)-1}k\right)|\nabla v|^{\alpha(x)-1}B(x)\cdot\nabla v + \left(|k|^{\alpha(x)-1}k\right)C(x)|v|^{\alpha(x)-1}v. \end{aligned} \tag{2.1}$$

A direct calculation yields

$$\nabla \cdot \left(|k|^{\alpha(x)-1}kA(x)|\nabla v|^{\alpha(x)-1}\nabla v\right) = \nabla\left(|k|^{\alpha(x)-1}k\right)\cdot\left(A(x)|\nabla v|^{\alpha(x)-1}\nabla v\right) + |k|^{\alpha(x)-1}k\nabla\cdot\left(A(x)|\nabla v|^{\alpha(x)-1}\nabla v\right). \tag{2.2}$$

Since

$$\nabla\left(|k|^{\alpha(x)-1}k\right) = |k|^{\alpha(x)-1}k(\log|k|)\nabla\alpha(x),$$

we obtain

$$\begin{aligned} \nabla \cdot \left(|k|^{\alpha(x)-1}kA(x)|\nabla v|^{\alpha(x)-1}\nabla v\right) &= A(x)|k|^{\alpha(x)-1}k(\log|k|)|\nabla v|^{\alpha(x)-1}\nabla\alpha(x)\cdot\nabla v \\ &\quad + |k|^{\alpha(x)-1}k\nabla\cdot\left(A(x)|\nabla v|^{\alpha(x)-1}\nabla v\right). \end{aligned} \tag{2.3}$$

Combining (2.1) with (2.3), we see that

$$(kv)Q[kv] = |k|^{\alpha(x)+1}vQ[v] \le 0$$

for any constant $k(\ne 0)$. Since $(kv)\log|kv| = 0$ for $k = 0$, we easily see that $(kv)Q[kv] = 0$ for $k = 0$. Therefore, we observe that (1.1) is half-linear. $\square$