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Positive solutions for a fourth order p-Laplacian boundary value problem[☆]

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ABSTRACT

In this paper, we study the existence, multiplicity and uniqueness of positive solutions for the fourth order p-Laplacian boundary value problem

$$\begin{cases} (|u''|^{p-1}u'')'' = f(t,u), \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad i = 0, 1. \end{cases}$$

Here $p > 0$ and $f \in C([0,1] \times \mathbb{R}^+, \mathbb{R}^+)$ ($\mathbb{R}^+ := [0, \infty)$). Based on a priori estimates achieved by utilizing properties of concave functions, we use fixed point index theory to establish our main results.

1. Introduction

In this paper, we study the existence, multiplicity and uniqueness of positive solutions for the fourth order p-Laplacian boundary value problem

$$\begin{cases} (|u''|^{p-1}u'')'' = f(t,u), & t \in (0,1), \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, & i = 0, 1, \end{cases} \tag{1.1}$$

where $p > 0$ and $f \in C([0,1] \times \mathbb{R}^+, \mathbb{R}^+)$ ($\mathbb{R}^+ := [0, \infty)$). Note that by a positive solution of (1.1) we mean a function $u \in C^2[0,1] \cap C^4(0,1)$ that solves (1.1) and satisfies $|u''|^{p-1}u'' \in C^2(0,1)$ and $u(t) > 0,\ t \in (0,1)$.

Second order differential equations with the p-Laplacian operator arise in modeling different physical and natural phenomena, which can be encountered in, for instance, non-Newtonian mechanics, nonlinear elasticity, glaciology, population biology, combustion theory, and nonlinear flow laws, see [1,2]. This explains why many papers have been published on existence of solutions for differential equations with the p-Laplacian operator; see, for instance, [3–14] and references therein. Fourth order boundary value problems, including those with the p-Laplacian operator, have their origin in beam theory [15,16], ice formation [17,18], fluids on lungs [19], brain warping [20,21], designing special curves on surfaces [22,20], etc. In beam theory, more specifically, a beam with a small deformation, a beam of a material which

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satisfies a nonlinear power-like stress and strain law and a beam with two-sided links which satisfies a nonlinear power-like elasticity law, can be described by fourth order differential equations along with their boundary value conditions. For more background and applications, we refer the reader to the work by Timoshenko [23] on elasticity, the monograph by Soedel [24] on deformation of structure, and the work by Dulácska [25] on the effects of soil settlement. Due to their wide applications, the existence and multiplicity of positive solutions for fourth order p-Laplacian boundary value problems has also attracted increasing attention over the last decades; see [26–40] and references therein.

The singular fourth-order p-Laplacian four-point boundary value problem

$$\begin{cases}(|u''|^{p-1}u'')'' = f(t, u(t)), \quad 0 < t < 1, \\ u(0) = u(1) - au(\xi) = u''(0) = u''(1) - bu''(\eta) = 0\end{cases} \tag{1.2}$$

arises in beam theory and was studied by Zhang and Liu [36], in which the method of upper and lower solutions was used in establishing their main results on positive solutions of the above problem. Here $p > 1, 0 < \xi, \eta < 1, f \in C((0, 1) \times (0, \infty), (0, \infty))$ may be singular at $t = 0$ and/or $t = 1$ and $u = 0$.

In [37], Zhang and Liu also considered the existence of positive solutions for (1.2), but with the original boundary value conditions in (1.2) replaced by the multi-point boundary conditions

$$u(0) - \sum_{i=1}^{m-2} a_i u(\xi_i) = u(1) = u''(0) - \sum_{i=1}^{m-2} b_i u''(\eta_i) = u''(1) = 0, \tag{1.3}$$

where $m \geq 3$, $a_i, b_i, \xi_i, \eta_i \in (0, 1)$ $(i = 1, 2, \ldots, m-2)$ are constants with $\sum_{i=1}^{m-2} a_i < 1$ and $\sum_{i=1}^{m-2} b_i < 1$, and $f \in C((0, 1) \times \mathbb{R}^+, \mathbb{R}^+)$ may be singular at $t = 0$ and/or $t = 1$. By virtue of monotone iterative techniques, they established a necessary and sufficient condition of the *pseudo*-$C^3[0, 1]$ positive solutions for their problem.

In [31], Guo et al. discussed the existence and multiplicity of positive solutions for the fourth-order quasilinear singular differential equation

$$(|u''|^{p-2}u'')'' = \lambda g(t) f(u), \quad 0 < t < 1, \tag{1.4}$$

sharing the boundary value conditions in (1.1), where λ is a positive parameter; $f \in C([0, \infty), (0, \infty))$ is nondecreasing on $[0, \infty)$, with $f(u) > \overline{\delta} u^m$ for all $u \in [0, \infty)$, where $\overline{\delta} > 0$ and $m > p - 1$ are two constants; $g \in C((0, 1), (0, \infty))$ and $g(t) \not\equiv 0$ on any subinterval of $(0, 1)$. Applying fixed point index theory and the method of upper and lower solutions, they proved that there is a threshold $\lambda^* < \infty$ for which (1.4) admits at least two positive solutions if $0 < \lambda < \lambda^*$, one positive solution if $\lambda = \lambda^*$, and no positive solution if $\lambda > \lambda^*$.

Our problem (1.1) merely involves the simply supported boundary value conditions and a nonsingular nonlinearity f. Nevertheless, our methodology and results in this paper are entirely different from those in the papers cited above. We observe that if $p = 1$, then (1.1) reduces to the Lidstone problem

$$\begin{cases}u^{(4)} = f(t, u), \quad t \in (0, 1), \\ u^{(2i)}(0) = u^{(2i)}(1) = 0, \quad i = 0, 1,\end{cases} \tag{1.5}$$

which can describe deformations of an elastic beam, with both ends simply supported, in an equilibrium state, and thus has been extensively studied (see [41–48] and references therein). It is natural to ask what connections *do* exist between (1.1) and (1.5). Some connections between them will be shown by repeatedly invoking Hölder's inequalities in our proofs. We will use fixed point index theory to establish our main results based on a priori estimates achieved by utilizing some properties of concave functions, properties including Hölder's inequalities and our new inequality (2.3) below.

This paper is organized as follows. Section 2 contains some preliminary results. Section 3 is devoted to the existence of positive solutions for (1.1) and Section 4 to the multiplicity of positive solutions for (1.1). In Section 5, we discuss the uniqueness of positive solutions for (1.1), and prove that the unique positive solution can be uniformly approximated by an iterative sequence beginning with any function u which is continuous, nonnegative and not identically vanishing on $[0, 1]$ (see Theorem 5.2). Section 6, finally, contains some remarks that serve to show some connections between (1.1) and (1.5).

2. Preliminaries

Let

$$E := C[0, 1], \qquad P := \{u \in E : u(t) \geq 0, \ \forall t \in [0, 1]\}, \qquad \|u\| := \max_{t \in [0,1]} |u(t)|.$$

Then $(E, \|\cdot\|)$ is a real Banach space and P a cone on E.

Let

$$G(t, s) := \begin{cases}t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1.\end{cases}$$