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Existence of solutions for a class of abstract differential equations with nonlocal conditions

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ABSTRACT

We discuss the existence of mild, classical and strict solutions for a class of abstract differential equations with nonlocal conditions. Our technical approach allows the study of partial differential equations with nonlocal conditions involving partial derivatives or nonlinear expressions of the solution. Some concrete applications to partial differential equations are considered.

1. Introduction

In this paper we study the existence of mild, classical and strict solutions for a class of abstract differential equations with nonlocal conditions of the form

$$u'(t) = Au(t) + f(t, u(t)), \quad t \in [0, a], \tag{1.1}$$

$$u(0) = g(u_{|_I}) + x_0 \in X, \tag{1.2}$$

where $A : D(A) \subset X \to X$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $(T(t))_{t\geq 0}$ defined on a Banach space $(X, \| \cdot \|)$, $x_0 \in X$, $I \subset (0, a]$, $u_{|_I}$ denotes the restriction of $u(\cdot)$ to I, $g \in C(C(I, X_\alpha), X)$, X_α ($\alpha \in (0, 1)$) is the domain of the fractional power $(-A)^\alpha$ of A endowed with the α-norm $\|x\|_\alpha = \|(-A)^\alpha x\|$ and $f : [0, a] \times X_\alpha \to X$ is a suitable function.

There exits extensive literature treating the existence and qualitative properties of solutions of abstract differential equations with nonlocal conditions. Concerning the motivations, relevant developments and the current status of the theory we refer the reader to [1–20] and the references therein.

The main novelty of this paper is the type of function $g(\cdot)$ considered in (1.2) and the associated applications. In this work, $g(\cdot)$ is a continuous function from $C(I, X_\alpha)$ into X which allows us to study partial differential equations with "nonlocal terms" involving partial derivatives or nonlinear expressions of the solution. The technical approach is based in the theory of analytic semigroups and our results and applications are totally new. It is interesting to note that in [2,10–12], the existence of mild solutions with values in X_α is studied for some abstract Cauchy problems with nonlocal conditions. However, the

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results in these papers are proved assuming that $g(\cdot)$ has values in X_α, which is a strong assumption and does not permit the described applications. We also note that the applications in [10,11] cannot be recovered as special cases of the abstract results since the functions which represent the nonlocal condition are really $L^2([0,1])$-valued.

This paper has three sections. In the next section, we study the existence of solutions for (1.1)–(1.2). In Theorem 2.1, we establish the existence and uniqueness of mild solutions for (1.1)–(1.2) via the contraction mapping principle. By using Schauder's fixed point Theorem and an approximation scheme, we also prove the existence of mild solutions for (1.1)–(1.2) (see Proposition 2.1, Lemmas 2.3, 2.4 and Theorem 2.2). It is important to note that Theorem 2.2 is proved without assuming some compactness conditions on $g(\cdot)$, which is a relevant fact in the theory of abstract differential equations with nonlocal conditions, see [13,16] for details. In the last part of Section 2 we discuss the existence of classical and strict solutions. Finally, in the last section, some concrete applications to partial differential equations with nonlocal conditions are presented.

Next, we include some notations and technicalities. Let $(Z, \|\cdot\|_Z)$ and $(W, \|\cdot\|_W)$ be Banach spaces. In this paper, $\mathcal{L}(Z, W)$ represents the space of bounded linear operators from Z into W endowed with the norm of operators denoted $\|\cdot\|_{\mathcal{L}(Z,W)}$, and we write $\mathcal{L}(Z)$ and $\|\cdot\|_{\mathcal{L}(Z)}$ when $Z = W$. In addition, $B_l(z, Z)$ denotes the closed ball with center at z and radius l in Z. As usual, for an interval $J \subset \mathbb{R}$, we use the notation $C(J, Z)$ for the space formed by all the continuous functions from J into Z endowed with the sup-norm denoted by $\|\cdot\|_{C(J,Z)}$.

In the remainder of this paper, we assume $0 \in \rho(A)$ ($\rho(A)$ is the resolvent set of A) and X_γ, $\gamma > 0$, is the domain of the fractional operator $(-A)^\gamma$ of A endowed with the norm $\|x\|_\gamma = \|(-A)^\gamma x\|$. From [21], we note the following result.

Lemma 1.1. *Let $\gamma, \beta \in (0, \infty)$. The space X_γ is a Banach space and $X_\gamma \hookrightarrow X_\beta$ when $0 < \beta < \gamma$. Moreover, the operator function $t \to (-A)^\beta T(t)$ belongs to $C((0,\infty), \mathcal{L}(X))$ and there exists $C_\beta > 0$ such that $\|(-A)^\beta T(t)\|_{\mathcal{L}(X)} \le C_\beta t^{-\beta}$ for all $t \in (0, a]$.*

2. Existence of solutions

In this section we study the existence of solutions for the nonlocal systems (1.1)–(1.2). To begin, we consider the following concepts of the solution.

Definition 2.1. A function $u \in C([0,a], X)$ is called a classical solution of (1.1)–(1.2) if $u \in C((0,a], X_1)$, the condition (1.2) is satisfied and $u(\cdot)$ is a solution of (1.1) on $(0, a]$.

Definition 2.2. A function $u \in C([0,a], X)$ is called a strict solution of (1.1)–(1.2) if $u \in C([0,a], X_1)$ and (1.1)–(1.2) are satisfied.

Definition 2.3. A function $u \in C([0,a], X)$ is said to be a mild solution of (1.1)–(1.2) if

$$u(t) = T(t)(x_0 + g(u_{|_I})) + \int_0^t T(t-s) f(s, u(s)) ds, \quad \forall t \in [0, a].$$

To prove our results, we introduce the following conditions. Next, q' denotes the conjugate of a number $q > 1$ and we take $q' = \infty$ for $q = 1$.

H_1 There are $\delta \in (0, a)$ and a nondecreasing function $W_g \in C([0,\infty), (0,\infty))$ such that $g \in C(C(I, X_\alpha), X)$, $g(u_{|_I}) = g(v_{|_I})$ for all $u, v \in C((0,a], X_\alpha)$ with $u_{|_I} = v_{|_I}$ and $\|g(u_{|_I})\| \le W_g(\|u_{|_{[\delta,a]}}\|_{C([\delta,a],X_\alpha)})$ for all $u \in C((0,a], X_\alpha)$.

H_2 For all $x \in X$, the function $f(\cdot, x)$ is strongly measurable on $[0, a]$ and $f(t, \cdot) \in C(X_\alpha, X)$ for each $t \in [0, a]$. There are $q \in \left[1, \frac{1}{\alpha}\right)$, $m_f \in L^{q'}([0,a], \mathbb{R}^+)$ and a non-decreasing function $W_f \in C([0,\infty), \mathbb{R}^+)$ such that $\|f(t,x)\| \le m_f(t) W_f(\|x\|_\alpha)$ for all $(t, x) \in [0, a] \times X_\alpha$.

H_3 The function $g(\cdot)$ belongs to $C(C(I, X_\alpha), X)$ and there are $\delta \in (0, a]$ and $L_g > 0$ such that $I \subset [\delta, a]$, $g(u_{|_I}) = g(v_{|_I})$ for all $u, v \in C((0,a], X_\alpha)$ with $u_{|_I} = v_{|_I}$ and

$$\|g(u_{|_I}) - g(v_{|_I})\| \le L_g \|u_{|_{[\delta,a]}} - v_{|_{[\delta,a]}}\|_{C([\delta,a],X_\alpha)}, \quad \forall u, v \in C((0,a], X_\alpha).$$

H_4 The function $f(\cdot)$ belongs to $C([0,a] \times X_\alpha, X)$ and there are $q \ge 1$ and $L_f \in L^{q'}([0,a], \mathbb{R}^+)$ such that

$$\|f(t,x) - f(t,y)\| \le L_f(t) \|x - y\|_\alpha, \quad \forall x, y \in X_\alpha, \ t \in [0, a].$$

Remark 2.1. In the remainder of this paper, we write simply $g(u)$ and u in place of $g(u_{|_I})$ and $u_{|I}$. In addition, $C_\alpha(X_\alpha)$ is the space

$$C_\alpha(X_\alpha) = \{u \in C((0,a], X_\alpha) : [|u|]_\alpha = \sup_{t \in (0,a]} t^\alpha \|(-A)^\alpha u(t)\| < \infty\}$$

endowed with the norm $[|u|]_\alpha$. It is easy to see that $C_\alpha(X_\alpha)$ is a Banach space.