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Study of multiplicity and uniqueness of solutions for a class of nonhomogeneous sublinear elliptic equations

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ABSTRACT

In this paper, we deal with the sublinear elliptic equation: $-\Delta u+V(x)u+a(x)|u|^p\, sgn(u) = f$ on $\mathbb{R}^N,\ N>2,\ 0<p<1$. Under suitable assumptions on the terms V and a, we prove some existence, uniqueness and multiplicity results. Continuity of solutions in the perturbation parameter f at 0 is also studied. Our main tools are the concentration-compactness principle and mountain-pass theorem.

1. Introduction

In this work, we are interested in the existence and uniqueness of solutions for the following problem

$$\begin{cases} -\Delta u + V(x)u + a(x)|u|^p\mathrm{sgn}(u) = f \\ u \in H^1(\mathbb{R}^N)\cap L^{p+1}(\mathbb{R}^N). \end{cases} \tag{1.1}$$

Our main hypotheses are cited below:

(H_1) $f \in L^2(\mathbb{R}^N)\cap L^{\frac{p+1}{p}}(\mathbb{R}^N),\ f\not\equiv 0,\ N>2,\ 0<p<1$
(H_2) $V\in L^\infty(\mathbb{R}^N)$, and $\lim_{|x|\to+\infty} V(x) = v_\infty \geq 0$
(H_3) $a \in L^\infty(\mathbb{R}^N)$, $\lim_{|x|\to+\infty} a(x) = a_\infty > 0$ and there exists $\alpha>0$ such that

$$a(x) > \alpha \quad \text{a.e. in } \mathbb{R}^N.$$

The superlinear equations $\left(\text{i.e. } 1<p<\frac{N+2}{N-2}\right)$ have been extensively studied by many authors. We can for instance cite [1–13]. Concerning our Eq. (1.1), similar ones have been studied in bounded domains with various boundary conditions (see [14–16] and references therein). In [16], KajiKiya proved the existence of infinitely many solutions of the following boundary value problem

$$\begin{cases} -\Delta u = |u|^p\mathrm{sgn}(u) + f(x,u) & x\in\Omega \\ u = 0 & x\in\partial\Omega \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ and $0<p<1$, under suitable assumptions on the nonlinearity term $f(\cdot,\cdot)$. For the case of unbounded domains and especially for the whole space $\mathbb{R}^N$, to the author's knowledge, very few results are known. We can, for example, quote [17,18]. In [17], Benrhouma and Ounaies proved the existence and the uniqueness of solutions of the following problem

$$-\Delta u + u = |u|^p\mathrm{sgn}(u) + f$$

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where $0 < p < 1$. In [18] Tehrani proved the existence of at least one solution of the following problem

$$-\Delta u + V(x)u = g(x, u) \quad x \in \mathbb{R}^N$$

where $g(\cdot, \cdot)$ is a sublinear function. (See also [19–21].)

In the present paper, we give sufficient conditions on V, a and f to have multiplicity of solutions of (1.1). We will show that for $\|f\|_2$ small enough and $f \geq 0$ a.e. in $\mathbb{R}^N$, the problem (1.1) admits a nonnegative solution which could be found through a minimizing process of some appropriately constructed functional. If the operator $-\Delta + V$ is positive i.e.

(H_4) $\int_{\mathbb{R}^N} (|\nabla\psi|^2 + V(x)\psi^2)dx \geq 0$ for every $\psi \in C_c^\infty(\mathbb{R}^N)$,

we prove the uniqueness of the solution, while if

(H_5) $\int_{\mathbb{R}^N} (|\nabla\varphi|^2 + V(x)\varphi^2)dx < 0$ for some $\varphi \in C_c^\infty(\mathbb{R}^N)$,

we can find a second solution by using the original version of the well known "mountain pass" theorem due to Ambrosetti, Rabinowitz [22]. If this second solution is not nonnegative, then we prove the existence of a third solution of (1.1) by applying the "mountain pass" theorem to an auxiliary problem depending on the first solution. The main result in the present work is given by the following theorem:

Theorem 1.1. *Assume* (H_1)–(H_3) *hold, then there exists a positive constant* c, *such that for every* $f \geq 0$, $\|f\|_2 < c$, *there exists a nonnegative solution* U_0 *of* (1.1). *Moreover:*

(1) *If* (H_4) *holds then, this solution is unique.*
(2) *If* (H_5) *holds then, the problem* (1.1) *admits at least a second solution* $V_0 \neq U_0$. *If, in addition,* V_0 *is not nonnegative, it follows the existence of a third solution.*

We divide this paper into five sections. In Section 2, we give some notations, preliminaries and useful results. In Sections 3–5 we prove Theorem 1.1. We should proceed in steps.

2. Notations and preliminaries

We will use the following notations:

(•) $\|u\|_q = \left(\int_{\mathbb{R}^N} |u|^q dx\right)^{\frac{1}{q}}$, $\forall\, 1 \leq q < \infty$
(•) c_s: the constant of Sobolev, Gagliardo, Nirenberg in $\mathbb{R}^N$,

$$\forall u \in H^1(\mathbb{R}^N), \quad \|u\|_{2^*} \leq c_s\|\nabla u\|_2 \text{ where } 2^* = \frac{2N}{N-2}$$

(•) $\text{sgn}(u)$: the sign of the function u.
(•) $F'(u)$: the Fréchet derivative of F at u.

Let

$$E = H^1(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N).$$

If we equip E with the norm

$$\|u\| = \|\nabla u\|_2 + \|u\|_{p+1}$$

it becomes a Banach reflexive space. Consider now the following functional,

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V|u|^2)dx + \frac{1}{p+1}\int_{\mathbb{R}^N} a(x)|u|^{p+1}dx - \int_{\mathbb{R}^N} fu dx.$$

It is clear that $I \in C^1(E, \mathbb{R})$ and any critical point of I is a weak solution of (1.1).

The following lemma is needed to study the existence of local minimum for the functional I.

Lemma 2.1. *Assume* (H_1)–(H_3) *hold, then there exist* $k > 0$, $\rho > 0$ *and* $L > 0$ *such that if* $\|f\|_2 < L$ *then* $I(u) \geq k$ *whenever* $\|u\| = \rho$.

Proof. Let $u \in E$, by Hölder and Sobolev inequalities there exists

$$r = \frac{(2^* - 2)(p+1)}{2(2^* - p - 1)} \in]0, 1[\quad \text{such that}$$

$$\begin{aligned}\|u\|_2^2 &\leq \|u\|_{p+1}^{2r}\|u\|_{2^*}^{2(1-r)} \leq c_s^{2(1-r)}\|u\|_{p+1}^{2r}\|\nabla u\|_2^{2(1-r)} \\ &\leq \frac{\alpha}{(p+1)(|V|_\infty + 1)}\|u\|_{p+1}^{p+1} + c_1\|\nabla u\|_2^q\end{aligned}$$