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Nonlinear Analysis 61 (2005) 839 – 855
Nonlinear Analysis
www.elsevier.com/locate/na

Blow-up solutions of inhomogeneous nonlinear Schrödinger equations on torus☆

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Received 1 September 2004; accepted 20 January 2005

Abstract

In this paper, we study blow-up solutions to the Cauchy problem of the inhomogeneous nonlinear Schrödinger equation

$$\partial_t u = i(f(x)\Delta u + \nabla f(x) \cdot \nabla u + k(x)|u|^2 u)$$

on $\mathbb{T}^2$. We present the L^2-concentration property for general initial data and investigate the L^2-minimality.

MSC: 35Q55; 35G25

Keywords: Inhomogeneous nonlinear Schrödinger equation; Blow-up

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☆ Supported in part by National University of Singapore Academic Research Fund Grant R-146-000-034-112; the third author (Y. Wang) is further supported in part by the National Key Basic Research Fund G1999075107 and the National Science Fund for Distinguished Young Scholars 10025104 of the People's Republic of China.

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doi:10.1016/j.na.2005.01.078

1. Introduction

Recently, Schrödinger-type equations with variable coefficients have been of considerable interest among both mathematicians and physicists, and some remarkable progress on the Cauchy problem has been made, see [5–7,11,12,14,15,22,24,28] and references therein for well-posedness, [18,23] for blow-up analysis, and [25] for Strichartz-type inequalities. In this paper we consider the following *inhomogeneous* cubic nonlinear Schrödinger equation (NLS) on the flat torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ (i.e., the space-periodic case):

$$\partial_t u = i(f(x)\Delta u + \nabla f(x) \cdot \nabla u + k(x)|u|^2 u), \tag{1.1}$$

together with the initial condition

$$u(0, \cdot) = u_0(\cdot), \tag{1.2}$$

where $f(x)$ and $k(x)$ are (non-constant) real-valued functions on $\mathbb{T}^2$. It is easy to see that the cubic nonlinearity is critical for blow-up in space-dimension two for focusing inhomogeneous NLS (while $|u|^{4/N}u$ for space-dimension *N*).

When the functions $f \equiv k \equiv 1$, the Cauchy problem (1.1)–(1.2) is the normal (homogeneous) critical NLS with space dimension two, which has been extensively studied by many mathematicians. The reader is referred to Refs. [2,9,27] for a comprehensive review for both non- and space-periodic cases, [3,8,13] for the bounded domains case (with Dirichlet boundary condition), and [19,20] for recent stunning progress on blow-up analysis. Antonini [1] studied the L^2-minimal blow-up solutions on $\mathbb{T}^N$. He showed the corresponding L^2-concentration property and gave a sharp lower bound of blow-up rate. His argument is based on the analysis on the local virial identity and the conservation of mass, energy and momentum.

For general real-valued functions $f(x)$ and $k(x)$, as in the homogeneous case, one can easily check that solutions of (1.1)–(1.2) obey conservation of mass and energy as follows:

$$\int_{\mathbb{T}^2} |u(t,x)|^2\,\mathrm{d}x = \int_{\mathbb{T}^2} |u_0(x)|^2\,\mathrm{d}x, \tag{1.3}$$

$$E_{f,k}(u(t)) = E_{f,k}(u_0), \tag{1.4}$$

where

$$E_{f,k}(u) = \frac{1}{2}\int_{\mathbb{T}^2} f(x)|\nabla u(x)|^2\,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{T}^2} k(x)|u(x)|^4\,\mathrm{d}x.$$

Momentum is no longer conserved in this case, however.

We first recall the following existence and uniqueness result for the problem (1.1)–(1.2):

Theorem 0 (*Pang et al. [22]*). *Let $s_0 \geqslant 4$ be an integer. Suppose $f \in C^{s_0+1}(\mathbb{T}^2)$ and $k \in C^{s_0}(\mathbb{T}^2)$ are real functions and $f(x)$ has no zero points. Then, given $u_0 \in H^{s_0}(\mathbb{T}^2)$, the Cauchy problem* (1.1)–(1.2) *admits a unique local smooth solution $u \in L^\infty([0,T), H^{s_0}(\mathbb{T}^2))$. Moreover, the solution is global in the sense that $u \in L^\infty_{\mathrm{loc}}([0,\infty), H^{s_0}(\mathbb{T}^2))$ provided $f \cdot k \leqslant 0$ or $\|u_0\|_{L^2}$ is small enough.*