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Existence and uniqueness of solutions of Abel integral equations with power-law non-linearities

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Abstract

In this article a class of nonlinear Abel-type integral equations is studied. These equations have at least the trivial solution, but we are interested in a continuous, positive solution. We show that there exists a unique positive solution in an order interval of a cone in a Banach space.

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1. Introduction

We study the following homogeneous Abel integral equation of the second kind with power-law nonlinearity:

$$u^{\mu}(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{K(x,s)\,u(s)}{(x-s)^{1-\alpha}}\,\mathrm{d}s, \quad x \in [0,T], \quad \mu > 1, \quad 0 < \alpha \leqslant 1, \tag{1}$$

where the function K is non-negative. Obviously, $u \equiv 0$ is the trivial solution of (1); however, the existence and uniqueness of solutions which are positive for $x > 0$ is of theoretical and practical interest. We note that the case $\alpha = 1$ is included in our considerations. In this case Eq. (1) is a Volterra equation, although not of Abel-type.

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A comprehensive reference on Abel-type equations, including an extensive list of applications, can be found in [13]. For further examples of applications, in particular for $\alpha = 1$, we refer to [23] (see also [22,25]). A special case of the above integral equation arises in heat transfer [20] and shock-wave propagation in gas-filled tubes [17]. Note that for $0 < \mu < 1$, Eq. (1) has only the trivial solution.

The question of the existence and uniqueness of solutions for this and related equations has been investigated by several authors by various methods of constructive and non-constructive type. Some results can be found in [3], a comprehensive reference on Volterra equations is [15] and on positive solutions to some differential and integral equations one can see [2]. A non-constructive proof of existence and uniqueness utilizes an Osgood-type condition; this method is used, e.g., in [14,21,24]. Constructive methods to show the existence of fixed points of monotone operators in cones of Banach spaces are given in [16,18]. Further references on the theory of cones in Banach spaces and especially non-linear operators acting on them are given by [4,10]. Usually, the uniqueness of the fixed point cannot be guaranteed by these methods, in contrast to the result of the contraction mapping theorem. To apply the latter theorem, one often constructs specialised spaces: this approach is used to obtain the existence and uniqueness of related integral equations in a metric space with Hilbert's projective metric in [5,6], and in spaces with suitable weighted norms or weighted metrics in [8,9,23].

In this article we will use an approach which is based on an analysis of the weighted metrics method. Problem (1) will be reformulated as a fixed-point equation, given on a subspace of the Banach space of continuous functions with supremum-norm, partially ordered by a cone. It will then be shown that this fixed-point operator is a contraction on an order interval; thus, the contraction mapping theorem can be applied. Related results can be found in [19,22,26]. In a sequel to this paper we will apply analogous considerations to discrete versions of the above integral equations. These can be interpreted as numerical approximations of the solution of (1). The framework developed in this and the subsequent article allows an analysis of these numerical methods.

2. Definitions and preliminary results

Here we provide precise conditions on the kernel function K. Eq. (1) has a convolution kernel, if we set $K(x,s) = k_1(x-s)$. For reference purposes we denote the explicitly non-convolution version of K by $k_2(x,s)$. The conditions to be imposed on the function K, or k_i, $i = 1, 2$, are as follows:

(a) $k_1 \in C^n[0,T]$, $n \in \mathbb{N}_0$, and $k_2 \in C([0,T]\times[0,T])$,
(b) $0 < k_1(x)\ \forall x \in (0,T]$ and $0 < k_2(x,s)\ \forall 0 \leqslant s \leqslant x \leqslant T$,
(c) $0 < \int_0^x K(x,s)\,(x-s)^{\alpha-1}\,\mathrm{d}s < \infty$ for all $x \in (0,T]$,
(d) $k_1(0) = k_1^{(1)}(0) = \ldots = k_1^{(n-1)}(0) = 0$, and $k_1^{(n)}(x) \geqslant k_1^{(n)}(0) > 0$, for all $x \in (0,T]$.

We will frequently apply the following estimate (the result for the special case of k_1 can be found in [23]):

$$\frac{1}{n!}\,(x-s)^n\,\mathbb{K}_{\mathrm{low}} \leqslant K(x,s) \leqslant \frac{1}{n!}\,(x-s)^n\,\mathbb{K}_{\mathrm{up}}, \quad 0 \leqslant s \leqslant x \leqslant T, \tag{2}$$