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Nonlinear Analysis 61 (2005) 559 – 575

Nonlinear Analysis

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Set valued functions in Fréchet spaces: Continuity, Hukuhara differentiability and applications to set differential equations

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Received 8 December 2004; accepted 3 January 2005

Abstract

The set $K_c(\mathbb{F})$ of compact convex subsets of a Fréchet space $\mathbb{F}$ is studied in detail and is realized as a projective limit of metric spaces. The notion of Hausdorff metric on it is replaced by a family of corresponding "semi-metrics" which provide the necessary background for the support of continuity and Lipschitz continuity. Finally the notion of Hukuhara derivative suited to our study is developed. The proposed approach forms the appropriate environment within which the study of set differential equations for Fréchet spaces can be developed. A first example on $\mathbb{R}^\infty$ is included.

MSC: primary 58B25; secondary 58A05

Keywords: Hukuhara differentiability; Set differential equations; Fréchet spaces

1. Introduction

The study of set (multivalued) differential equations saw a rapid development over the last few years because of the advantages that this theory provides: One has only to work

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0362-546X/$ - see front matter
doi:10.1016/j.na.2005.01.004

with a semilinear complete metric space instead of a complete normed linear space needed in the study of ordinary differential equations. On the other hand, the latter are obtained as a special case of set differential equations (SDE) if we use single valued mappings.

The notion of Hukuhara derivative plays a fundamental role in the theory of SDE (see [2,4]): Given a multifunction $F : I \to K_c(\mathbb{R}^n)$, where I is an interval of real numbers and $K_c(\mathbb{R}^n)$ the space of all nonempty compact and convex subsets of $\mathbb{R}^n$, we demand the existence of an element $D_H F(t_0) \in K_c(\mathbb{R}^n)$, $t_0 \in I$, such that the limits

$$\lim_{\Delta t \to 0+} \frac{F(t_0 + \Delta t) - F(t_0)}{\Delta t}, \quad \lim_{\Delta t \to 0+} \frac{F(t_0) - F(t_0 - \Delta t)}{\Delta t}$$

both exist and are equal to $D_H F(t_0)$. The differences used in the above limits are the Hukuhara differences of the involved sets.

Another notion of basic importance for the study of SDE is the Hausdorff distance between two nonempty sets A and B of $\mathbb{R}^n$:

$$D(A, B) = \max\{d_H(A, B), d_H(B, A)\},$$

where d_H stands for the Hausdorff separation of A from B:

$$d_H(A, B) = \sup\{d(a, B); a \in A\},$$

and $d(a, B)$ for the distance from the point a to the set B.

Continuity as well as Lipschitz continuity of multivalued functions are also based on the Hausdorff distance.

The above methodology (described in detail in [4], and briefly presented in the next section) can also be adapted to the framework of infinite-dimensional Banach spaces. However, if we take one step further and try to extend it to Fréchet (i.e. Hausdorff, metrizable and complete locally convex topological vector) spaces, most of the previous notions do not extend in a routine manner.

More precisely, one easily sees that the Hausdorff distance cannot be defined at all, since the topology of a Fréchet space is not obtained by a single norm. As a result, the definition of continuity of set valued mappings seems to collapse. This is also the case for Lipschitz continuity. On the other hand, the notion of Hukuhara differentiation also needs to be revised.

Despite the above-mentioned serious obstacles, the study of SDE in non-Banach locally convex topological vector spaces is a subject of main importance since several problems of modern analysis, differential geometry and theoretical physics reduce to differential equations on this type of spaces. It is important to note that in this framework there is not any general solvability theory for differential equations—even for linear ones: An ordinary differential equation may admit no or multiple solutions for the same initial condition. There is, therefore, an increased need for alternative ways of studying differential equations within the Fréchet context and the theory of set differential equations seems to be an appealing and promising one.

In this paper, after presenting the preliminaries in Section 2, we set up the appropriate framework within which the study of SDE can be developed for a Fréchet space $\mathbb{F}$. The set $K_c(\mathbb{F})$ of compact convex subsets of $\mathbb{F}$ is studied in detail and is realized by a new convenient