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Blowup behavior for a nonlinear parabolic equation of nondivergence form

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1. Introduction

We consider the following Cauchy problem for a nonlinear parabolic equation:

$$u_t = u(u_{xx} + u^p), \quad x \in (-\infty, \infty), t > 0, \tag{1.1}$$

$$u(x, 0) = u_0(x), \quad x \in (-\infty, \infty), \tag{1.2}$$

where $p > 1$ and $u_0(x) > 0$. We say that a solution $u(x, t)$ blows up in a finite time T if $\sup_{x\in(-\infty,\infty)} u(x, t) \to \infty$ as $t \to T$.

It is well known that the solution to the system (1.1)–(1.2) blows up in a finite time T for certain $u_0(x)$. Suppose that u blows up at $(0, T)$. To describe the blowup behavior near the point $(0, T)$, we introduce the following similarity variables:

$$y = \frac{x}{(T-t)^\beta}, \tag{1.3}$$

$$T - t = e^{-s}, \tag{1.4}$$

$$z(y, s) = (T-t)^\alpha u(x, t), \tag{1.5}$$

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where the similarity exponents are necessarily given by

$$\alpha = \frac{1}{p}, \quad \beta = \frac{p-1}{2p}.$$

Then u satisfies (1.1)–(1.2) if and only if z satisfies

$$z_s + \alpha z + \beta y z_y = z(z_{yy} + z^p), \quad y \in (-\infty, \infty), s > s_0 := -\ln T, \tag{1.6}$$

$$z(y, s_0) = z_0(y) := T^\alpha u_0(T^\beta y), \quad y \in (-\infty, \infty). \tag{1.7}$$

Hence to study the blowup behavior of u near the blowup point $(0, T)$ is equivalent to study the stabilization problem of (1.6) as $s \to \infty$. Note that the only nonzero constant solution of (1.6) is given by

$$\kappa := \alpha^\alpha = (1/p)^{1/p}. \tag{1.8}$$

Motivated by a work of Galaktionov [2], we shall describe the blowup behavior of the solution near the blowup point by constructing a Lyapunov function (see, e.g., [7]). To construct a suitable Lyapunov function, we need to study the stationary solution of (1.6):

$$\varphi_{yy} - \beta y \varphi^{-1} \varphi_y + \varphi^p - \alpha = 0. \tag{1.9}$$

Actually, to overcome the difficulties of possible singularity near the origin, we shall modify the ODE (1.9) near the origin. For such a modification, we also refer to [5] for a study on a quenching problem. The main difference between these two problems is the term in which the singularity appears. The singularity appears in the reaction term for the quenching problem in [5]. Here, we need to modify the singularity in the gradient term for our blowup problem.

We define the ω-limit set of the problem (1.6)–(1.7) by

$$\omega(z_0) = \{\varphi \in C^2(\mathbb{R}) \mid \text{ there exists a sequence } s_j \to \infty \text{ such that}$$
$$z(\cdot, s_j) \to \varphi(\cdot) \text{ as } j \to \infty \text{ uniformly on compact sets in } \mathbb{R}\}.$$

We state our main theorems as follows:

Theorem 1.1. *Assume that $u_0(x)$ is smooth and*

$$\begin{array}{ll} u_0(x) > 0 & \text{for } x \in (-\infty, \infty), \\ u_0''(x) + u_0^p(x) \geqslant 0 & \text{for } x \in (-\infty, \infty), \\ u_0(x) = u_0(-x) & \text{for } x > 0, \\ u_0'(x) \leqslant 0 & \text{for } x > 0, \ u_0'(x) \not\equiv 0, \end{array}$$

and that T is the blowup time. Then the ω-limit set of (1.6)–(1.7) *is not empty and any ω-limit is a solution of the ODE* (1.9) *such that*

$$\varphi_\xi \leqslant 0, \quad \varphi_{\xi\xi} + \varphi^p \geqslant 0 \quad \text{for } \xi \in [0, \infty).$$

Theorem 1.2. *If we further assume*

$$\liminf_{x \to \infty} x^{\alpha/\beta} u_0(x) = +\infty, \tag{1.10}$$