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The asymptotic behavior of the composition of two resolvents

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Abstract

The asymptotic behavior of the composition of two resolvents in a Hilbert space is investigated. Connections are made between the solutions of associated monotone inclusion problems and their dual versions. The applications provided include a study of an alternating minimization procedure and a new proof of von Neumann's classical result on the method of alternating projections.

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1. Introduction

Throughout, $\mathscr{H}$ is a real Hilbert space with inner product $\langle \cdot \mid \cdot \rangle$ and induced norm $\|\cdot\|$. Let A and B be two maximal monotone operators from $\mathscr{H}$ to $2^{\mathscr{H}}$ with resolvents J_A and J_B, respectively, and let $\gamma \in \,]0, +\infty[$. Our paper is concerned with the inclusion problem

$$\text{find } (x, y) \in \mathscr{H}^2 \quad \text{such that} \quad (0, 0) \in (\mathrm{Id} - R + \gamma(A \times B))(x, y), \tag{1}$$

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where $R: (x, y) \mapsto (y, x)$. This abstract formulation subsumes a wide spectrum of problems in nonlinear analysis and its applications. We thus recover problems arising in variational inequalities [30], best approximation [12], image processing [4,44], mechanics [34], and optimization [1,31]. The dual inclusion problem associated with (1) is

$$\text{find } (x^*, y^*) \in \mathscr{H}^2 \quad \text{such that} \\ (0,0) \in ((\mathrm{Id} - R)^{-1} + (A^{-1} \times B^{-1}) \circ (\mathrm{Id}/\gamma))(x^*, y^*). \tag{2}$$

Now consider the alternating resolvent method

$$x_0 \in \mathscr{H} \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad y_n = J_{\gamma B} x_n, \quad x_{n+1} = J_{\gamma A} y_n, \tag{3}$$

where $\mathbb{N} = \{0, 1, 2, \ldots\}$. The objective of the present paper is to provide a systematic investigation of the asymptotic behavior of the sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$, $(y_n - x_n)_{n\in\mathbb{N}}$, and $(x_{n+1} - y_n)_{n\in\mathbb{N}}$ generated by this algorithm in connection with the solutions of (1) and (2). When specialized to the case when A and B are subdifferentials, our results will be significantly refined and will yield new insights into an alternating minimization procedure.

The remainder of the paper is organized as follows. Section 2 contains basic notation and auxiliary results on nonexpansive and monotone operators. In Section 3, we provide a detailed investigation of the asymptotic behavior of (3). The applications discussed in that section include variational inequalities as well as the problem of finding cycles for inconsistent feasibility problems. In Section 4, the results of Section 3 are sharpened in the context of proximity operators and we obtain new results on the primal and dual behavior of an alternating minimization procedure. Among the applications presented is a new proof of von Neumann's classical result on the convergence of alternating projections.

2. Auxiliary results

We recall some useful results on monotone operators and resolvents. Let $A: \mathscr{H} \to 2^{\mathscr{H}}$ be a set-valued operator. The sets $\operatorname{dom} A = \{x \in \mathscr{H} \mid Ax \neq \varnothing\}$, $\operatorname{ran} A = \{u \in \mathscr{H} \mid (\exists\, x \in \mathscr{H})\ u \in Ax\}$, and $\operatorname{gr} A = \{(x, u) \in \mathscr{H}^2 \mid u \in Ax\}$ are the domain, the range, and the graph of A, respectively. The inverse of A is the set-valued operator A^{-1} with graph $\{(u, x) \in \mathscr{H}^2 \mid u \in Ax\}$, the resolvent of A is $J_A = (\mathrm{Id} + A)^{-1}$, and the Yosida approximation of A of index $\gamma \in \,]0, +\infty[$ is

$${}^{\gamma}\!A = (\mathrm{Id} - J_{\gamma A})/\gamma = (\mathrm{Id} + A^{-1}/\gamma)^{-1} \circ (\mathrm{Id}/\gamma). \tag{4}$$

The operator A is monotone if $\langle x - y \mid u - v \rangle \geqslant 0$, for all (x, u) and (y, v) in $\operatorname{gr} A$. If A is monotone and $\operatorname{gr} A$ cannot be enlarged without destroying monotonicity, then A is maximal monotone. A classical result due to Minty [35] implies that A is maximal monotone if and only if J_A is firmly nonexpansive with domain $\mathscr{H}$. We now provide basic properties of firmly nonexpansive operators (see [24, Sections 1.9 and 1.11] for proofs and additional properties).