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Nonlinear Analysis 61 (2005) 1041 – 1056
Nonlinear Analysis
www.elsevier.com/locate/na

On resonance Hamiltonian systems without the Palais–Smale condition

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Received 1 November 2002; accepted 30 November 2002

Abstract

In this paper, we study periodic solutions of resonance Hamiltonian systems. By combining Conley index theory and Galerkin approximation method, we prove the existence of periodic solutions of Hamiltonain systems where both the Palais–Smale condition and the strong resonance method fail.

MSC: 58E05; 58F05; 34C25

Keywords: Periodic solution; Hamiltonian systems; Conley index; Maslov-type index; Galerkin approximation

1. Introduction

In this paper we study 1-periodic solutions of the following Hamiltonian system

$$\dot{z} = JH'(t, z), \tag{1.1}$$

where $H'(t, z)$ denotes the gradient of $H(t, z)$ with respect to the z variable,

$$J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$$

is the standard $2N \times 2N$ symplectic matrix, and N is a positive integer. We denote by (x, y) and $|x|$, the usual inner product and norm in $\mathbb{R}^{2N}$, respectively. We assume the following conditions for H.

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0362-546X/$ - see front matter
doi:10.1016/j.na.2002.11.001

(H1) $H \in C^2([0,1] \times \mathbb{R}^{2N}, \mathbb{R})$ is a 1-periodic function in t, and satisfies

$$|H''(t,z)| \leqslant a_1|z|^s + a_2, \quad \forall (t,z) \in \mathbb{R} \times \mathbb{R}^{2N}, \text{ where } s \in (1,\infty), \ a_1, a_2 > 0;$$

(H2) $|H'(t,z) - B_\infty(t)z|$ is bounded for all $(t,z) \in \mathbb{R} \times \mathbb{R}^{2N}$;
(H3) $|H'(t,z) - B_0(t)z| = o(|z|)$ as $|z| \to 0$ uniformly in t.

Here $B_0(t)$ and $B_\infty(t)$ are $2N \times 2N$ symmetric matrices, continuous and 1-periodic in t. Condition (H2) implies that system (1.1) is an asymptotically linear Hamiltonian system.

The existence of periodic solutions of (1.1) has been studied by many authors (cf. [1–6,8–20] and the references therein). When (H2) holds, there are mainly two type conditions on

$$G_\infty(t,z) = H(t,z) - \tfrac{1}{2}(B_\infty(t)z, z).$$

One type is the Landesman–Lazer-type condition, i.e.,

$$G_\infty(t,z) \to \pm\infty \quad \text{as } |z| \to +\infty. \tag{1.2}$$

In this case, the Palais–Smale condition is satisfied. Many variational methods can be used to find periodic solutions of (1.1) (cf. [16,5,15]). Another type is the strong resonance condition, i.e.,

$$G_\infty(t,z) \to 0, \quad G'_\infty(t,z) \to 0 \quad \text{as } |z| \to +\infty. \tag{1.3}$$

In this case, the Palais–Smale condition fails. But one can apply the strong resonance method developed by Chang [5] to get the periodic solutions (cf. [4,11,6]).

In this paper we shall study the existence of 1-periodic solutions of system (1.1) in cases that both the Palais–Smale condition and the strong resonance method fail. We assume the following conditions for $G_\infty(t,z)$.

($\mathrm{H4}^{\pm}$) There exist $c_1, c_2 > 0$, $L > 0$ and $\alpha > 0$ such that for $|z| \geqslant L$

$$\pm(G'_\infty(t,z), z) \geqslant c_1/|z|^\alpha; \quad |G'_\infty(t,z)||z| \leqslant c_2|(G'_\infty(t,z), z)|.$$

According to [8,14,13], for a given continuous 1-periodic and symmetric matrix function $B(t)$, one can assign a pair of integers $(i,n) \in \mathbb{Z} \times \{0,\ldots,2N\}$ to it, which is called the Maslov-type index of $B(t)$. We denote by (i_0, n_0) and (i_∞, n_∞) the Maslov-type indices of $B_0(t)$ and $B_\infty(t)$ respectively. Our first result reads as:

Theorem 1.1. *Suppose that H satisfies* (H1)–(H3). *Then system* (1.1) *possesses a nontrivial* 1-*periodic solution if one of the following two cases occurs*:

(i) ($\mathrm{H4}^{+}$) *and* $i_\infty + n_\infty \notin [i_0, i_0 + n_0]$.
(ii) ($\mathrm{H4}^{-}$) *and* $i_\infty \notin [i_0, i_0 + n_0]$.

Remark 1.2. (1) The Palais–Smale condition may not be satisfied in Theorem 1.1. One can see examples in Section 4 for more details.