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Strongly nonlinear parabolic equations with natural growth terms in Orlicz spaces

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Abstract

We prove approximation and compactness results in inhomogeneous Orlicz-Sobolev spaces and look at, as an application, the Cauchy-Dirichlet equation $u' + A(u) + g(x, t, u, \nabla u) = f \in W^{-1,x}E_M$, where A is a Leray-Lions operator having a growth not necessarily of polynomial type. We also give a trace result allowing to deduce the continuity of the solutions with respect to time.

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1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ and let Q be the cylinder $\Omega \times (0, T)$ with some given $T > 0$ and let

$$A(u) = -\mathrm{div}(a(x, t, u, \nabla u))$$

be a Leray–Lions operator defined on $L^p(0, T; W^{1,p}(\Omega))$.

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Landes–Mustonen [20] and Dall'aglio–Orsina [8] proved the existence of solutions for the following parabolic initial-boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} + A(u) + g(x,t,u,\nabla u) = f & \text{in } Q, \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where g is a nonlinearity with the following "natural" growth condition (of order p):

$$|g(x,t,s,\xi)| \leqslant b(|s|)(c(x,t) + |\xi|^p)$$

and which satisfies the classical sign condition $g(x,t,s,\xi)s \geqslant 0$. The right-hand side f is assumed to belong to $L^{p'}(0,T;W^{-1,p'}(\Omega))$.

This result generalizes analogous ones of Lions [21], Landes [18] when $g \equiv 0$ and of Brézis–Browder [7], Landes–Mustonen [19] for $g \equiv g(x,t,u)$. See also [4,5] for related topics. In these results, the function a is supposed to satisfy a polynomial growth condition with respect to u and ∇u.

In the case where a satisfies a more general growth condition with respect to u and ∇u, it is shown in [9] that the adequate space in which (1) can be studied is the inhomogeneous Orlicz–Sobolev space $W^{1,x}L_M(Q)$ where the N-function M is related to the actual growth of a. The solvability of (1) in this setting is proved by Donaldson [9] for $g \equiv 0$ and by Robert [22] for $g \equiv g(x,t,u)$ when A is monotone, $t^2 \ll M(t)$ and $\overline{M}$ satisfies a Δ_2 condition and also by Elmahi [11] for $g = g(x,t,u,\nabla u)$ when M satisfies a Δ' condition and $M(t) \ll t^{N/(N-1)}$ as application of some L_M compactness results in $W^{1,x}L_M(Q)$, see [10].

In a recent work, the authors [12] have established an existence result for problems of the form (1), when $g \equiv 0$, without assuming any restriction on the N-function M.

It is our purpose in this paper to prove the existence of solutions for problem (1) in the setting of Orlicz spaces for general N-functions M with a nonlinearity $g(x,t,u,\nabla u)$ having natural growth with respect to the gradient.

We first prove (cf. Section 3), an approximation theorem (Theorem 1) which allows to regularize an arbitrary test function by smooth ones with converging time derivatives and which be also applied to get a trace result giving the continuity of such test function with respect to time.

Next, in Section 4, we establish L^1-compactness results in the inhomogeneous Sobolev space $W^{1,x}L_M(Q)$, nearly similar to those of Simon [23] and Boccardo–Murat [5] and Elmahi [10], and which, however, are sufficient to deal with approximate equations. Section 2 will be devoted to some preliminaries concerning Orlicz spaces.

Note that our existence result generalizes analogous ones of [8,20]. Moreover, and contrary to [8,20], the proof is achieved without extending the initial problem or assuming the vanishing of the initial condition u_0.

2. Preliminaries

2.1. Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, i.e. M is continuous, convex, with $M(t) > 0$ for $t > 0$, $M(t)/t \to 0$ as $t \to 0$ and $M(t)/t \to \infty$ as $t \to \infty$.