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Perturbations and Hopf bifurcation of the planar discontinuous dynamical system

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Abstract

The objective of the paper is to obtain results on the behavior of a specific plane discontinuous dynamical system in the neighbourhood of the singular point. A new technique of investigation is presented. Conditions for existence of the foci and centres are proposed. The focus-centre problem and Hopf bifurcation are considered. Appropriate examples are given to ilustrate the bifurcation theorem.

Keywords: Planar dynamical systems; Discontinuous systems; Focus; Centre; Stability; Hopf bifurcation

1. Introduction and Preliminaries

The theory of dynamics with discontinuous trajectories has been developed through applications [3,4,6,15,19,20,22,30,32,34,35] and theoretical challenges [15,16,22,26,36–38]. The present paper can be considered as an attempt to apply ideas of the perturbations theory, which was founded by Poincaré and Lyapunov [28,33], and methods of the bifurcation theory [4,8,15,18,21,29,30,33] to the object which combines features of vector fields and maps. In fact, we consider the problem for equations with variable time of impulses. Effective methods of the investigation of systems with nonfixed moments of impulsive action can be found in [9,12,13,26,27,38]. Theoretical problems of nonsmooth dynamics and

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discontinuous maps [5,10,11,14,23–25] also very close to the subject of our paper. In our view there have been two principal obstacles to a thorough investigation of the subject. While the absence of sufficiently general results on the smoothness of solutions has been the first one, the problem of choice of a nonperturbed system convenient for study has been the second. The present work utilizes extensively the differentiable and analytical dependence discontinuous solutions on parameters [1,2,26,40]. Moreover, the nonperturbed equation is specifically defined. And, while all of its terms are linear, this equation is essentially a nonlinear one. We should remark that in general the perturbed systems with sets of discontinuities of linear nature have been considered previously. One example is the clock model [22,38]. But we investigate systems all of whose terms are nonlinear. This necessitates the use of the standard linearization method and application of the concept of B-equivalent discontinuous systems which has been developed in [1–3] for nonautonomous case. The approach of the paper can be effectively employed for investigation of oscillations in mechanics, electronics, biology and medicine [4,15,30–32,34].

The paper is organized in the following manner. In Section 1, we give the description of the systems under consideration and prove the theorem of existence of foci and centres of the nonperturbed system. The main subject of Section 2 is foci of the perturbed equation. The noncritical case is considered. In Section 3 the problem of distinguishing between the centre and the focus is solved. Bifurcation of a periodic solution is investigated in Section 4. Section 5 consists of examples illustrating the bifurcation theorem.

1.1. *The nonperturbed system*

Let N, R be sets of all natural and real numbers, respectively, R^2 be a real euclidean space. Denote by $\langle x, y\rangle$ the dot-product of vectors $x, y \in R^2$. Let $||x|| = \langle x, x\rangle^{\frac{1}{2}}$ be the norm of a vector $x \in R^2$, $\mathscr{R}$ be the set of all real-valued constant 2×2 matrices, $I \in \mathscr{R}$ be an identity matrix. We shall consider in R^2 the following dynamical system:

$$\begin{aligned}&\frac{\mathrm{d}x}{\mathrm{d}t} = Ax, \quad x \notin \Gamma_0,\\ &\Delta x|_{x\in\Gamma_0} = B_0 x,\end{aligned} \tag{1}$$

where $A, B_0 \in \mathscr{R}$, Γ_0 is a subset of R^2 and will be described below. The phase point of (1) moves between two consecutive intersections with the set Γ_0 along one of the trajectories of the system $x' = Ax$. When the solution meets the set Γ_0 at the moment τ, the point $x(t)$ has an jump $\Delta x|_\tau := x(\tau+) - x(\tau)$. Thus we suppose that the solutions are left continuous functions. The following assumptions will be needed throughout the paper:

(C1) $\Gamma_0 = \bigcup_{i=1}^{p} s_i$, $p \in N$, where s_i are half-lines starting at the origin and defined by equations $\langle a^i, x\rangle = 0$, $i = \overline{1, p}$, where $a^i = (a_1^i, a_2^i) \in R^2$ are constant vectors;

(C2)
$$A = \begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix},$$

where $\beta \neq 0$;