Available online at www.sciencedirect.com
SCIENCE DIRECT®

Nonlinear Analysis 60 (2005) 179–185

Nonlinear Analysis

www.elsevier.com/locate/na

Positive solutions of nonlinear differential equations with prescribed decay of the first derivative

Octavian G. Mustafa[a, b, *]

[a] *Department of Mathematics, University of Craiova, Al. I. Cuza 13, Craiova, Romania*
[b] *Department of Mathematics, Lund University, P.O. Box 118, SE-22100 Lund, Sweden*

Received 1 July 2004; accepted 24 August 2004

Abstract

An existence and uniqueness result for bounded, positive solutions $x(t)$ of the equation $u'' + f(t, u, u') = 0$, $t \geqslant t_0 \geqslant 0$, is established by means of the Banach contraction principle. For such a solution it is shown that $\alpha(t) \leqslant x'(t) \leqslant \beta(t)$, $t \geqslant t_0$, where α, β are given nonnegative, continuous functions which are integrable over $[t_0, +\infty)$. The result complements others known in the literature.

MSC: 34A34; 34E05; 47H10

Keywords: Monotone positive solution; Nonlinear differential equation; Banach contraction principle

1. Introduction

We are concerned here with the existence and uniqueness of solutions to the problem

$$u'' + f(t, u, u') = 0, \quad t \geqslant t_0 \geqslant 0, \tag{1}$$

$$u(t) > 0, \quad t \geqslant t_0, \quad \lim_{t \to +\infty} u(t) = M \in (0, +\infty), \tag{2}$$

$$\alpha(t) \leqslant u'(t) \leqslant \beta(t), \quad t \geqslant t_0, \tag{3}$$

where f, α, β are cont inuous functions satisfying certain hypotheses that will be introduced later.

* Corresponding author.
E-mail address: octaviangenghiz@yahoo.com (O.G. Mustafa).

0362-546X/$ - see front matter
doi:10.1016/j.na.2004.08.032

Condition (2), or similar, has been used successfully to obtain nonoscillation results for Eq. (1). This topic was richly documented over the years as the reader can see by consulting the list of references in [15,20]. Also, it proved to be of use in the existence theory for positive solutions of semilinear elliptic problems in exterior domains (see [3,4,21,22] and the references therein).

The terminal value problem included in (1)–(3), that is,

$$\begin{cases} u'' + f(t,u,u') = 0, & t \geqslant t_0 \geqslant 0, \\ \lim\limits_{t\to+\infty} u(t) = M \in \mathbb{R}, \end{cases}$$

has a long history as part of the general asymptotic integration theory of ordinary differential equations (see [1,6,7,9,10,13] and the references therein). However, specific particular cases, such as the one under investigation here, should be studied separately according to [19, p. 388]. The same problem was investigated recently in connection with Weyl's limit circle/limit point classification of differential operators in the theory of singular Sturm–Liouville problems (see [16–18]).

Condition (3), where α and β are nonnegative L^1- functions, seems to have been neglected by researchers since most papers on this subject refer to either

$$u'(t) = o(t^{-1}) \quad as\ t \to +\infty \tag{4}$$

(see [7, Theorem 2], [8, Theorem 3], [13, Corollary 1]) or

$$u'(t) = o(1) \quad as\ t \to +\infty$$

(see [11,14,15]). An investigation of solutions to Eq. (1) satisfying condition (3) can be regarded, in our opinion, as complementary to studies such as the one by Kiguradze and Kvinikadze [12].

The recent paper by Dubé and Mingarelli [5] contains an ingenious modification of the method of Hale and Onuchic [7, Eq. (1)] which, combined with the application of classical Banach contraction principle, leads to an existence and uniqueness result for bounded positive solutions of the nonlinear differential equation

$$u'' + f(t,u) = 0, \quad t \geqslant t_0 \geqslant 0.$$

In this note, by using a different modification of the Hale–Onuchic technique, we derive an existence and uniqueness result for problem (1)–(3). It should be mentioned here that the uniqueness results, with or without Lipschitz-like assumptions on $f(t,u,u')$, are of special interest in asymptotic integration theory (see [10, Theorem 6.1] or [2, Theorem]).

2. Existence of positive monotone solutions

Theorem 1. *Let $M \in \mathbb{R}$ be fixed and α, $\beta \in L^1((t_0,+\infty);\mathbb{R})$ be two continuous nonnegative functions such that $\alpha(t) \leqslant \beta(t)$ for all $t \geqslant t_0$ and* $\lim_{t\to+\infty}\alpha(t) = \lim_{t\to+\infty}\beta(t) = 0$.