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Meshless Least-Squares Method for Solving the Steady-State Heat Conduction Equation*

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Abstract: The meshless weighted least-squares (MWLS) method is a pure meshless method that combines the moving least-squares approximation scheme and least-square discretization. Previous studies of the MWLS method for elastostatics and wave propagation problems have shown that the MWLS method possesses several advantages, such as high accuracy, high convergence rate, good stability, and high computational efficiency. In this paper, the MWLS method is extended to heat conduction problems. The MWLS computational parameters are chosen based on a thorough numerical study of 1-dimensional problems. Several 2-dimensional examples show that the MWLS method is much faster than the element free Galerkin method (EFGM), while the accuracy of the MWLS method is close to, or even better than the EFGM. These numerical results demonstrate that the MWLS method has good potential for numerical analyses of heat transfer problems.

Key words: meshless; least-squares; heat conduction; steady-state

Introduction

In the past twenty years, a series of numerical methods called meshless methods (also called meshfree methods) have been developing rapidly. The first meshless method was smoothed particle hydrodynamics developed by Lucy[1] and Gingold and Monaghan[2] in 1977, and then thoroughly studied by Monaghan[3]. After the element free Galerkin method (EFGM) was proposed by Belytschko et al. in 1994[4], meshless methods have drawn more and more attention and have been successfully applied to various problems in solid mechanics, fluid mechanics, heat transfer, and electromagnetic fields[5-7].

Most kinds of meshless methods have been built upon discretization schemes like the Galerkin method, the Petrov-Galerkin method, or the direct collocation method. Generally speaking, meshless methods of the Galerkin and Petrov-Galerkin types need numerical integration, which results in much more computational effort than the finite element method (FEM) in most cases; while the direct collocation meshless method suffers from instabilities.

Like the least-squares finite element method (LSFEM)[8], meshless methods can also be based on least-square schemes. The meshless weighted least-squares (MWLS) method[9] is such a method. Application of MWLS to elastostatics and wave propagation problems has shown that it is accurate, stable, and efficient.

In this paper, the MWLS method is extended to solve heat conduction problems. The basic MWLS formulation for solving steady-state heat conduction problems is developed, and the optimal choice of computational parameters is discussed. Several 2-D examples are presented with the numerical results compared with analytical and EFGM solutions.

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1 Moving Least-Squares Approximation

In the moving least-squares (MLS) scheme, the local approximation of the field variable $u(\boldsymbol{x})$ is expressed as

$$u(\boldsymbol{x}) \approx u^h(\boldsymbol{x}, \bar{\boldsymbol{x}}) = \sum_{i=1}^{m} p_i(\bar{\boldsymbol{x}}) a_i(\boldsymbol{x}) = \boldsymbol{p}^{\mathrm{T}}(\bar{\boldsymbol{x}}) \boldsymbol{a}(\boldsymbol{x}) \quad (1)$$

where $p_i(\bar{\boldsymbol{x}})$ is the basis function, generally a complete monomial, m is the number of terms in the basis function, and $a_i(\boldsymbol{x})$ are the coefficients, which are determined by minimizing the following L_2 norm,

$$J = \sum_I w_I(\boldsymbol{x}) \left(u^h(\boldsymbol{x}, \boldsymbol{x}_I) - u(\boldsymbol{x}_I) \right)^2 = \sum_I w_I(\boldsymbol{x}) \left[\boldsymbol{p}^{\mathrm{T}}(\boldsymbol{x}_I) \boldsymbol{a}(\boldsymbol{x}) - u_I \right]^2 \quad (2)$$

where u_I is the value of $u(\boldsymbol{x})$ at node $\boldsymbol{x}_I$, and $w_I(\boldsymbol{x})$ is the weight function that is usually a compactly supported function which is only nonzero in a small neighborhood called the “support domain” of node $\boldsymbol{x}_I$ where it reaches its maximum value. Many kinds of weight functions have been used in meshless methods. The cubic spline function is used in this paper,

$$w(r) = \begin{cases} 2/3 - 4r^2 + 4r^3, & r \le 1/2; \\ 4/3 - 4r + 4r^2 - 4r^3/3, & 1/2 < r \le 1; \\ 0, & r > 1 \end{cases} \quad (3)$$

where r is the normalized radius equal to the ratio of the distance between node I and the evaluation point to the radius of the support domain.

The minimization of the function J is equivalent to

$$\frac{\partial J}{\partial \boldsymbol{a}} = \boldsymbol{A}(\boldsymbol{x}) \boldsymbol{a}(\boldsymbol{x}) - \boldsymbol{B}(\boldsymbol{x}) \boldsymbol{u} = \boldsymbol{0} \quad (4)$$

where the matrices are given by

$$\boldsymbol{A}(\boldsymbol{x}) = \sum_{I=1}^{n} w_I(\boldsymbol{x}) \boldsymbol{p}(\boldsymbol{x}_I) \boldsymbol{p}^{\mathrm{T}}(\boldsymbol{x}_I),$$

$$\boldsymbol{B}_I(\boldsymbol{x}) = w_I(\boldsymbol{x}) \, \boldsymbol{p}(\boldsymbol{x}_I) \quad (5)$$

$$\boldsymbol{u} = \left(u_1, \quad u_2, \quad \cdots, \quad u_n \right)^{\mathrm{T}} \quad (6)$$

Substituting the coefficients $\boldsymbol{a}(\boldsymbol{x})$ from Eq. (4) into Eq. (1), the MLS approximation can be expressed as

$$u^h(\boldsymbol{x}) = \sum_{I=1}^{n} N_I(\boldsymbol{x}) \, u_I \quad (7)$$

where the shape function $N_I(\boldsymbol{x}) = \boldsymbol{p}^{\mathrm{T}}(\boldsymbol{x}) \boldsymbol{A}^{-1}(\boldsymbol{x}) \boldsymbol{B}_I(\boldsymbol{x})$.

2 Basic Equations for Heat Conduction Problems and the Least-Squares Discretization

The steady-state temperature distribution in domain $\varOmega$ is governed by

$$k \nabla^2 u(\boldsymbol{x}) + \rho Q = 0, \qquad \boldsymbol{x} \in \varOmega \quad (8)$$

with the boundary conditions:

$$u = \bar{u}, \qquad \boldsymbol{x} \in \varGamma_1 \quad (9)$$

$$\boldsymbol{n} \bullet k \nabla u = \bar{q}, \qquad \boldsymbol{x} \in \varGamma_2 \quad (10)$$

$$\boldsymbol{n} \bullet k \nabla u = h\left(u_{\mathrm{a}} - u \right), \qquad \boldsymbol{x} \in \varGamma_3 \quad (11)$$

where k and ρ represent the thermal conductivity and the density, Q is the heat source per unit mass. $\bar{u}(\boldsymbol{x})$ is the prescribed temperature, and $\bar{q}(\boldsymbol{x})$ is the prescribed heat flux. h denotes the convection heat-transfer coefficient, $u_{\mathrm{a}}(\boldsymbol{x})$ is the prescribed ambient temperature, and $\boldsymbol{n}$ represents the unit outward normal to the boundary.

If the field variable is approximated by the MLS scheme in Eq. (7), Eq. (8) and the boundary conditions Eqs. (9)-(11) cannot be satisfied exactly, which leads to residuals. Different ways to minimize the residuals correspond to different discretization schemes, such as the Galerkin method, the Petrov-Galerkin method, and the direct collocation method, all of which can be regarded as special cases of the weighted residual method[10]. In this paper, the residuals are minimized in a least-squares manner, as the sum of the squares of the residuals,

$$\varPi = \int_{\varOmega} R^2(\boldsymbol{x}) \mathrm{d}\varOmega + \int_{\varGamma_1} \alpha_1 \bar{R}_1^2(\boldsymbol{x}) \mathrm{d}\varGamma + \int_{\varGamma_2} \alpha_2 \bar{R}_2^2(\boldsymbol{x}) \mathrm{d}\varGamma + \int_{\varGamma_3} \alpha_3 \bar{R}_3^2(\boldsymbol{x}) \mathrm{d}\varGamma \quad (12)$$

which is to be minimized. $R(\boldsymbol{x})$ and $\bar{R}_i(\boldsymbol{x})$ refer to the residuals corresponding to the governing equation and the boundary conditions on $\varGamma_i$ ($i = 1, 2, 3$) and α_i is the weight coefficient which is used as a penalty to enforce the boundary conditions. In Eq. (12), the function $\varPi$ is an integral, which requires numerical quadrature in the final equations and increases the computational effort. To overcome this shortcoming, the following discrete functional is used instead.