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A new strain energy function for the hyperelastic modelling of ligaments and tendons based on fascicle microstructure

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A R T I C L E I N F O

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A B S T R A C T

A new strain energy function for the hyperelastic modelling of ligaments and tendons based on the geometrical arrangement of their fibrils is derived. The distribution of the crimp angles of the fibrils is used to determine the stress–strain response of a single fascicle, and this stress–strain response is used to determine the form of the strain energy function, the parameters of which can all potentially be directly measured via experiments – unlike those of commonly used strain energy functions such as the Holzapfel–Gasser–Ogden (HGO) model, whose parameters are phenomenological. We compare the new model with the HGO model and show that the new model gives a better match to existing stress–strain data for human patellar tendon than the HGO model, with the average relative error in matching this data when using the new model being 0.053 (compared with 0.57 when using the HGO model), and the average absolute error when using the new model being 0.12 MPa (compared with 0.31 MPa when using the HGO model).

1. Introduction

Ligaments and tendons are fundamental structures in the musculoskeletal systems of vertebrates. Ligaments connect bone to bone to provide stability and allow joints to function correctly, whereas tendons connect bone to muscle to allow the transfer of forces generated by muscles to the skeleton. The wide variety of roles played by different ligaments and tendons requires them to have considerably different mechanical responses to applied forces, and their differing stress–strain behaviours have been well documented (Benedict et al., 1968; Tipton et al., 1986).

Ligaments and tendons consist of collagenous fibres organised in a hierarchical structure (Kastelic et al., 1978). Their main subunit is the fascicle, which consists of fibrils arranged in a crimped pattern (see Fig. 1). Further subunits in the hierarchy can be observed; however, the mechanics on these lengthscales will not be considered here. Instead, we shall focus on the effect of the geometrical arrangement of the fibrils within fascicles on the stress–strain properties of ligaments and tendons.

From a modelling perspective, ligaments and tendons can be categorised as fibre-reinforced biological soft tissues. A wide variety of models has been proposed to describe such tissues; however, to the author's knowledge, there has not yet been a successful attempt to develop a constitutive model within a non-linear elastic framework that includes the required anisotropy and characteristic stress–strain behaviour, which is non-linear with increasing stiffness for small strains (this region of the stress–strain curve is commonly termed the *toe region*) and subsequently linear, and, crucially, depends *only on directly measurable parameters*. Existing models are either phenomenological (Fung, 1967; Holzapfel et al., 2000), or lacking in the required material properties (such as the neo-Hookean model, which was developed for modelling rubber, but has still been used extensively in modelling biological soft tissues Miller, 2001, 2005), or both (Gou, 1970).

Early work on modelling biological tissue was carried out by Fung (1967). Fung showed that the stress in rabbit mesentery under uniaxial tension appears to increase exponentially as a function of increasing stretch. This exponential stress–strain relationship appears to approximate the behaviour of many biological soft tissues well, but only in a phenomenological sense and there is no microstructural basis for the choice of the exponential function. In 1970, Gou built upon Fung's work and proposed an isotropic strain energy function (SEF) for biological tissues that similarly gives an exponential stress–strain relationship in the case of uniaxial tension, but since this model is isotropic, it is not suitable for modelling anisotropic tissues such as ligaments and tendons.

With regard specifically to ligaments and tendons, various models were proposed over the following decades, as summarised in the review article by Woo et al. (1993). The models proposed involved infinitesimal elasticity (Frisen et al., 1969), finite elasticity (Hildebrant

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Nomenclature

W	strain energy function
c, k_1, k_2	material parameters of Holzapfel–Gasser–Ogden model
I_1, I_2	isotropic strain invariants
I_4	anisotropic strain invariant
$\mathbf{B}, \mathbf{C}$	left/right Cauchy–Green tensor
$\mathbf{M}, \mathbf{m}$	direction of fascicles in undeformed/deformed configuration
$\mathbf{F}$	deformation gradient tensor
c_i	phenomenological material parameters
T	$c_2(I_1-3)^2+c_3(I_4-1)^2+c_4(I_1-3)(I_4-1)$
$\tilde{a}$	fascicle radius
$\tilde{\rho}, \rho$	dimensional/non-dimensional radial variable in fascicle
$\theta_p(\rho)$, $\hat{\theta}_p(\rho)$	fibril crimp angle distributions
θ_o, θ_i	crimp angle of outermost/innermost fibrils
α, p	crimp angle distribution parameters
$\epsilon_p(\rho)$	strain in fascicle as fibrils at radius ρ become taut
ϵ^*	strain in fascicle as outer fibrils become taut
b	crimp blunting factor
ϵ	given strain in fascicle
R_p	radius within which all fibrils are taut for a given ϵ
P_p	tensile loads experienced by fascicle
$\sigma_p(\rho)$, $\epsilon_p^f(\rho)$	stress/strain in fibrils at radius ρ
E^*	Hooke's law parameter utilised by Kastelic et al. (1980)
$\Delta\epsilon_p(\rho)$	"elastic deformation" at radius ρ
E	Young's modulus of fibrils
$l_p(\rho)$, L	initial fibril/fascicle length
$\Delta l_p(\rho)$, ΔL	fibril/fascicle extension
τ_p, $\hat{\tau}_p$	average tractions in the direction of the fascicle
λ	stretch in the direction of the fascicle
β	$2(1-\cos^3\theta_o)/(3\sin^2\theta_o)$
$\mathbf{T}$, $\mathbf{T}^{HGO}$	Cauchy stresses
J	det $\mathbf{F}$
Q, Q^{HGO}	Lagrange multipliers
ϕ	fibre volume fraction
$\mathbf{T}_f$, $\mathbf{t}_f$	component of stress/traction associated with fascicles
$\hat{\mathbf{m}}$	unit vector in direction of $\mathbf{m}$
γ, η	constants of integration, defined in Eq. (51)
μ	ground state shear modulus of ligament/tendon matrix
R, Θ, Z	coordinate variables in undeformed configuration
r, θ, z	coordinate variables in deformed configuration
A, a	undeformed/deformed radius of ligament/tendon
B, b	undeformed/deformed length of ligament/tendon
ζ	stretch in longitudinal direction of ligament/tendon
$\mathbf{E}_R$, $\mathbf{E}_\Theta$, $\mathbf{E}_Z$	basis vectors in undeformed configuration
$\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$	basis vectors in deformed configuration
$\mathbf{n}$	outer unit normal to curved surface of ligament or tendon
S_{zz}, S_{zz}^{HGO}	longitudinal nominal stresses
e	engineering strain
δ, δ^{HGO}	relative errors
Δ, Δ^{HGO}	absolute errors

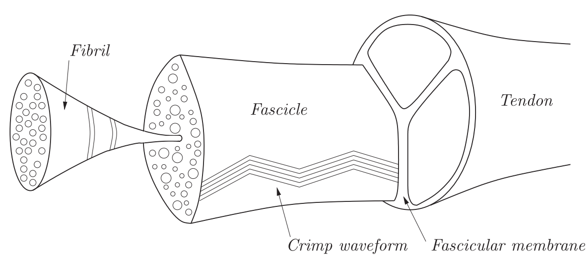

Fig. 1. Tendon hierarchy (adapted from Kastelic et al., 1978).

et al., 1969), quasi-linear viscoelasticity (Fung, 1968) and single integral finite strain viscoelasticity theory (Johnson et al., 1992). In particular, we note the work of Kastelic et al. (1980), in which a model was developed for the stress–strain response of a fascicle, taking into account the distribution of fibril crimp. It was shown that a radial variation in the crimp angle of a fascicle's fibrils could lead to a non-linear stress–strain relationship of the form typically observed in tension tests. Unfortunately, however, an error in the implementation of Hooke's law in that paper led to the derived relationship being incorrect, as we discuss further in Section 2.

Arguably the most influential model to be developed in the last 20 years for modelling biological tissues is the SEF proposed by Holzapfel et al. (2000), often referred to as the Holzapfel–Gasser–Ogden (HGO) model:

$$W=\frac{c}{2}(I_1-3)+\frac{k_1}{k_2}\left(e^{k_2(I_4-1)^2}-1\right), \tag{1}$$

where I_1 and I_4 are strain invariants, defined by

$$I_1=\mathrm{tr}\,\mathbf{C}, \quad \text{and} \quad I_4=\mathbf{M}\cdot(\mathbf{CM}), \tag{2}$$

where $\mathbf{C}=\mathbf{F}^T\mathbf{F}$ is the right Cauchy–Green tensor, where $\mathbf{F}$ is the deformation gradient tensor (Ogden, 1997), and $\mathbf{M}$ is a unit vector pointing in the direction of the tissue's fibres before any deformation has taken place, c, k_1 and k_2 are material parameters, and the above expression is only valid when $I_4 \geq 1$ (when $I_4<1$, $W=c/2(I_1-3)$). This SEF was proposed as a constitutive model for arteries and is commonly used, along with its variants (Holzapfel and Ogden, 2010) to model a wide variety of biological materials. The advantages of this model are clear – it retains an elegant mathematical simplicity, whilst also providing the required anisotropy and "exponential-shaped" stress–strain curve common to many biological materials; however, as it is a phenomenological model, the parameters c, k_1 and k_2 cannot be directly linked to measurable quantities, and therefore the model has restricted predictive capabilities.

A large number of phenomenological, transversely isotropic, non-linear elastic models of biological soft tissues have been proposed. The following models were collated by Murphy (2013), where the parameters c_i, $i=1,2,3,4,5,6,7$ are material parameters that can be chosen to match experimental data. Humphrey and Lin (1987) proposed this strain energy function for modelling passive cardiac tissue:

$$W=c_1\left(e^{c_2(I_1-3)}-1\right)+c_3\left(e^{c_4(I_4^{1/2}-1)^2}-1\right). \tag{3}$$

Humphrey et al. (1990) proposed the following for passive myocardium:

$$W=c_1(I_4^{1/2}-1)^2+c_2(I_4^{1/2}-1)^3+c_3(I_1-3)+c_4(I_4^{1/2}-1)(I_1-3)+c_5(I_1-3)^2. \tag{4}$$

Fung et al. (1993) proposed

$$W=c_1(e^T-T-1)+c_5(I_1-3)^2+c_6(I_4-1)^2+c_7(I_1-3)(I_4-1), \tag{5}$$