Journal of Financial Economics 114 (2014) 600–612

Contents lists available at ScienceDirect

Journal of Financial Economics

journal homepage: www.elsevier.com/locate/jfec

Advancing the universality of quadrature methods to any underlying process for option pricing ☆

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ARTICLE INFO

Article history:
Received 25 April 2013
Received in revised form
14 October 2013
Accepted 3 November 2013
Available online 2 August 2014

JEL classification:
G13
C63

Keywords:
Universal quadrature
QUAD
Option pricing
Numerical techniques
Transition density function

ABSTRACT

Exceptional accuracy and speed for option pricing are available via quadrature (Andricopoulos, Widdicks, Duck, and Newton, 2003), extending into multiple dimensions with complex path-dependency and early exercise (Andricopoulos, Widdicks, Newton, and Duck, 2007). However, the exposition is incomplete, leaving many modelling processes outside the Black-Scholes-Merton framework unattainable. We show how to remove the remaining major block to universal application. Although this had appeared highly problematic, the solution turns out to be conceptually simple and implementation is straightforward (we provide code on the Journal of Financial Economics website at http://jfe.rochester.edu). Crucially, the method retains its speed and flexibility across complex combinations of option features but is now applicable across other underlying processes.

1. Introduction

Numerical techniques are widely required in derivatives pricing, since it is often the case that no analytic equation has been found for the valuation of a particular class of option. Ideally, in place of numerical methods, we would eventually have a suite of analytic solutions to cover all derivatives pricing situations or, failing that, analytic approximations of sufficient accuracy and utility for all practical cases. For example, the work of Kristensen and Mele (2011) is highly encouraging, yet we remain a long way from generality along this route. Beyond the solutions of Black and Scholes (1973) and Merton (1973) and a limited set of other cases (generally those with no early exercise), numerical techniques are frequently required. The available numerical techniques are classified as trees (Cox, Ross, and Rubinstein, 1979), solution of partial differential equations usually by finite difference methods starting with the most basic explicit method (Brennan and Schwartz, 1977), Monte Carlo simulation (Boyle, 1977) and quadrature in the form of the QUAD technique (Andricopoulos, Widdicks, Duck, and Newton, 2003).

Each of these has been the subject of modification and refinement, especially in relation to handling early exercise with Monte Carlo (Longstaff and Schwartz, 2001) and path-dependent features with the other techniques. Andricopoulos, Widdicks, Newton, and Duck (2007) further developed QUAD into a flexible, robust option pricing tool of wide applicability, covering multiple dimensions, early exercise and heavy path-dependence in complex combinations of exercise features. QUAD is usually overwhelmingly fast, making it especially useful in those cases in which computation with other methods is inconveniently slow. However, it has largely been limited to the Black-Scholes-Merton framework. Overcoming

☆ David P. Newton thanks the Cambridge Judge Business School, University of Cambridge, UK, for facilities provided during a visiting fellowship in 2013.
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http://dx.doi.org/10.1016/j.jfineco.2014.07.014
0304-405X/

this limitation is the subject of our paper, which completes the exposition of the method.

Just as the mathematics of trees, finite difference and Monte Carlo approaches were all known and used in the natural sciences and engineering long before their introduction into finance, basic quadrature goes back centuries. In essence, it is the calculation of an area under a graph via an approximation, splitting the area into a series of shapes, such as rectangles, and summing their individual areas. Taking smaller shapes produces more accurate results, converging on the correct one. Well-known methods for doing this are the Trapezium Rule, Simpson's Method and Gaussian quadrature, and there are others. Each has differing properties and is more or less easy to program, but of particular interest is the rate of convergence to a correct solution as the number of calculations is increased in progressively finer approximations.

A key concept in the financial application of quadrature, sometimes not appreciated, is that the mathematical quadrature component is merely a computational engine to be chosen appropriately to fit into the wider calculations of the particular options problem (Andricopoulos, Widdicks, Duck, and Newton, 2003; Andricopoulos, Widdicks, Newton, and Duck, 2007). Thus, even the very simple Trapezium Rule can be adequate when elements in the wider calculations are less refined. Similarly, Gaussian quadrature, though in itself a very fast scheme, only provides useful extra speed over what may be the best practical engine, Simpson's Method, where unusually heavy calculational demands are made on the quadrature component versus the rest of the computational scheme. We shall return to the engine analogy later, when we show how previously intractable problems in applying quadrature can be circumvented by including a second type of numerical "engine".

The foundation work was presented in the Black-Scholes-Merton framework but (as explained in Section 2.2) the technique applies whenever the conditional probability density function is known. This restricts the immediate use of the technique to the Black-Scholes-Merton setup, to Merton's jump-diffusion model (Merton, 1976) and to certain interest rate models such as those of Vasicek (1977) and Cox, Ingersoll, and Ross (1985). Extension to Merton's process is straightforward. The interest rate models are more subtle, though Heap (2008) has successfully extended the coverage to some (but not all) interest rate derivatives with mean-reverting underlying processes.

A notable advance was made by O'Sullivan (2005), who used the observation that many useful processes without a well-known density function do, nonetheless, have a well understood characteristic function. The density function, as the inverse Fourier transform (FFT) of the characteristic function, can be computed using fast Fourier transform and the output may then be inserted in the QUAD scheme to price derivatives. We refer to this method as FFT-QUAD. O'Sullivan's method applies in particular to exponential Levy processes. This made FFT-QUAD an important advance but it does suffer several drawbacks. First, it requires two integrations even for a derivative on a single underlying process. This brings the complexity of the algorithm to at least $O(N^2)$, where N is the number of grid points used in the numerical integrations; by comparison, the original QUAD has a much better complexity of just $O(N)$ for vanilla options. Second, it does not cover every option type; for example, the single-variable FFT-QUAD cannot be used to price heavily path-dependent options in stochastic volatility frameworks, since it does not keep track of the evolution of the volatility process in moving from one observation point to the next.

O'Sullivan's FFT-QUAD was improved considerably by the CONV technique of Lord, Fang, Bervoets, and Oosterlee (2007). We refer to this method as CONV-QUAD (Staunton, 2007). This excellent method uses the observation that the fundamental pricing integral may usually be regarded as the convolution (strictly speaking, the cross-correlation) of the payoff and the density function. The beauty of this insight is that the two integrals of FFT-QUAD may then be replaced by two fast Fourier transforms. This brings the complexity of the algorithm down to $O(N\log(N))$ and, for example, for Bermudan options (on M observation points), the complexity remains at $O(MN\log(N))$, which beats even QUAD's $O(MN^2)$. The CONV-QUAD method applies to exponential Levy processes and, hence, in particular to the Black-Scholes-Merton model, thereby improving on the speed and accuracy of the plain QUAD technique of Andricopoulos, Widdicks, Duck, and Newton (2003) and Andricopoulos, Widdicks, Newton, and Duck (2007). Due to its nearly linear speed, it clearly replaces plain QUAD as the fastest method for a great many cases.

Useful as these developments were, the road to full universality for underlying processes remains blocked. The CONV method cannot be applied to, for example, the CEV or the Heston processes with early exercise and, while a single-variable characteristic function for the latter has been used in O'Sullivan (2005) and Fang and Oosterlee (2008) to price European options, a universal QUAD-style treatment of these processes is still lacking.

In this paper we return to the methods of Andricopoulos, Widdicks, Duck, and Newton (2003) and Andricopoulos, Widdicks, Newton, and Duck (2007) and provide option pricing techniques for the missing underlying processes. At the core of this extension is the use of closed-form approximations for the appropriate single- or two-variable transition density functions. By using these approximations we can price complex combinations of option features precisely as if we were working in the Black-Scholes-Merton framework. Thus, we advance the range of the earlier papers without losing their generality; the universality promised in the title of the first paper (Andricopoulos, Widdicks, Duck, and Newton, 2003) is finally arrived at.

2. Basics

Descriptions of the QUAD method can be found in Andricopoulos, Widdicks, Duck, and Newton (2003), Andricopoulos, Widdicks, Newton, and Duck (2007) and Chen (2013). We also provide a detailed appendix on the Journal of Financial Economics website (http://jfe.rochester.edu).

2.1. QUAD in the Black-Scholes-Merton framework

Start with the well-known Black-Scholes-Merton partial differential equation for an option with an underlying