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Stability of the replicator equation with continuous strategy space

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Abstract

We extend previous work that analyzes the stability of evolutionary dynamics on probability distributions over continuous strategy spaces. The stability concept considered is that of "neighborhood" convergence to a rest point (i.e. an equilibrium distribution over the strategy space) under the dynamics in the weak topology for all initial distributions whose support is close to that of the rest point. Stability criteria involving strategy domination and neighborhood superiority are developed for monomorphic rest points (i.e. the equilibrium distribution is supported on a single strategy) and for distributions that have finite support.

Keywords: Convergence; Coevolution; Replicator dynamics; Monomorphism; Weak topology

JEL classification: C72; C70

1. Introduction

Evolutionary dynamics for continuous strategy spaces have received considerable attention recently both among theoretical biologists who are interested in the coevolution of species traits and among economists who concentrate instead on predicting rational behavior of individuals whose payoffs are given through game interactions. Most

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theoretical research on stability for coevolutionary models (e.g. Abrams, 2001; Doebeli and Dieckmann, 2000; Marrow et al., 1996 and the references therein) make the simplifying assumption that each species is monomorphic (or homogeneous) and remains so during the course of evolution (i.e. all individuals present in a given species exhibit the same behavior) and that the population is at its equilibrium size for the current monomorphism.[1] This leads to the stability analysis of what are known as adaptive or strategy dynamics. Although there has been much less research in this area from the economic or game-theoretic perspective (e.g. Bomze, 1990, 1991; Seymour, 2000; Oechssler and Riedel, 2001, 2002), this latter literature typically considers the full dynamical system where aggregate behavior is described by a distribution on the space S of individual strategy choices and assume individual payoffs are defined in terms of a function f on $S \times S$.

In this paper, we follow the latter approach applied to a symmetric game with a continuous strategy space. These references in the economic literature spend a great deal of time developing the evolutionary dynamics on the set of probability distributions (e.g. the replicator equation), proving its solutions are well-defined, and relating its properties to static equilibrium conditions (that generalize those for the case where there are a finite set of strategies that may be used by the population as in the matrix games of Section 2.1). We benefit from their work by briefly summarizing this development at the beginning of Section 2 and devoting the remainder of the paper to analyzing the stability of equilibrium distributions (i.e. rest points) for the replicator equation.

Immediate issues that arise in this analysis are what constitutes closeness and/or convergence for probability distributions and for what initial distributions we expect this convergence. The main problem is that there are several ways to define these topological concepts that generalize the accepted approach when there are a finite set of strategies. Moreover, as clearly demonstrated by Oechssler and Riedel (2002), conclusions concerning stability depend critically on which definitions are taken. These issues are clarified in Section 2.1 by referring to well-known stability results for matrix games and, in the process, motivate our stability concept (see Definition 1 at the beginning of Section 3).

Our main goal is to derive conditions on f that predict stability. Section 3 completely characterizes (Proposition 1 and Theorem 1) stability of monomorphic rest points (i.e. probability distributions concentrated at a single strategy) and relates these results to the coevolutionary literature for monomorphic populations. Section 4 extends these results to probability distributions with finite support, giving sufficient conditions for stability (Theorems 2 and 3). Several examples are developed to illustrate the theorems. Examples 1a, 1b and 1c in Sections 2, 3 and 4 respectively apply the theory to a model based on Cournot competition with n identical firms that have a constrained maximum output. These latter two examples illustrate respectively situations where the evolutionary dynamics predicts all firms will have the same production level and where some firms produce at maximum output while others produce nothing. Examples 2 and 3 are based on functions f taken from the adaptive dynamics literature, demonstrating in turn Theorem 1

[1]Notable exceptions are Vincent et al. (1996) and Cressman and Garay (2003) where polymorphic effects and the effects of varying population size are also taken into account. See also Example 2 in Section 3.