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A Markov chain approach to renormalization group transformations

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ABSTRACT

We aim at an explicit characterization of the renormalized Hamiltonian after decimation transformation of a one-dimensional Ising-type Hamiltonian with a nearest-neighbor interaction and a magnetic field term. To facilitate a deeper understanding of the decimation effect, we translate the renormalization flow on the Ising Hamiltonian into a flow on the associated Markov chains through the Markov–Gibbs equivalence. Two different methods are used to verify the well-known conjecture that the eigenvalues of the linearization of this renormalization transformation about the fixed point bear important information about all six of the critical exponents. This illustrates the universality property of the renormalization group map in this case.

1. Introduction

The discovery of the equivalence of Markov random fields and Gibbs random fields was a major breakthrough in the interchange of ideas between probability and physics. A Markov random field is a natural generalization of the familiar concept of a Markov chain, which is a collection of random variables with the property that, given the present, the future is (conditionally) independent of the past. If we look at the chain itself as a very simple graph and ignore the directionality implied by "time", then a Markov chain may alternatively be viewed as a chain graph of stochastic variables, where each variable is independent of all other variables (both future and past) given its two neighbors. A Markov random field is the same thing, only that rather than a chain graph, we allow the relationship between the variables to be defined by any graph structure, and each variable is independent of all the others given its neighbors in the graph. A Gibbs random field, on the other hand, is formed by a set of random variables whose configurations obey a Gibbs distribution, which is a probability distribution that factorizes over all possible cliques, i.e. complete subgraphs in the graph, and the factors are conveniently referred to as "clique potentials". These two ways of defining a random configuration are apparently quite different [1]: A Markov random field is characterized by its local property (the Markovianity) whereas a Gibbs random field is specified by its global property (the Gibbs distribution).

The rigorous study of the relationship between these two seemingly unrelated fields was initiated by Dobrushin [2] in the context of statistical physics, who considered the questions of existence and uniqueness of a random field subject to a Markovian conditional distribution. Further investigations quickly ensued. Averintsev [3] and Spitzer [4] independently proved that the class of two-state Markov chains is identical to the class of Gibbs ensembles on the simple cubic lattice. Hammersley and Clifford [5] showed that the same equivalence holds between a multi-state Markov field and a generalized Gibbs ensemble over an arbitrary finite graph. The celebrated Hammersley–Clifford theorem states that each Markov field with a system of neighbors and the associated system of cliques is also a Gibbs field with the same system of cliques, and vice versa, each Gibbs field is also a Markov field with the corresponding system of neighbors. This implies that the joint

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probability and the conditional probability can specify each other, and serves as a theoretical basis for many modeling applications, where the global characteristic is captured and represented through a set of tractable local characteristics. The original method of proof, however, did not have great intuitive appeal, and many alternative proofs of this theorem were developed. Sherman [6] verified the equivalence of Markov fields and Gibbs ensembles under more relaxed conditions by the repeated use of the inclusion–exclusion principle. Preston [7] adopted a direct approach to the two-state problem and presented an explicit formula for the pair potential. Grimmett [8] showed that the equivalence of structure follows immediately from an application of the Möbius inversion theorem. A final improvement was done by Besag [9], who applied methods of statistical analysis and gave a much simpler, analytical proof of the general result.

The nearest-neighbor Ising model in one dimension is commonly used to demonstrate the powerful Markov–Gibbs equivalence. Though an ordered phase only emerges at zero temperature, this classic model is physically important in that it has a fixed point (the so-called "zero temperature phase transition") where the critical exponents may be sensibly defined as in higher dimensions. There is the astonishing empirical fact that these critical exponents depend only on overall features of the system, and are related to the eigenvalues of the linearized renormalization group map near the fixed point [10]. This universality conjecture has generated continued interest in the scientific community, and various approaches to the renormalization effect on the one-dimensional Ising model have been explored [11,12].

Consider a one-dimensional Ising model with N spins $\sigma_i = \pm 1$, labeled successively $i = 0, \dots, N-1$. We take the system size N to be very large (strictly speaking, infinite). The Gibbs field of this model is described by a Hamiltonian H, consisting of a nearest-neighbor interaction J and a magnetic field term m:

$$H = -\left(J\sum_{i=0}^{N-1}\sigma_i\sigma_{i+1} + m\sum_{i=0}^{N-1}\sigma_i\right), \tag{1}$$

where periodic boundary condition is imposed so that $\sigma_N = \sigma_0$, a standard setup to ensure that H is translation-invariant. We focus on a specific renormalization group transformation, namely decimation transformation with blocking factor b. To avoid unnecessary technicalities, we assume that b divides N. The decimation procedure is straightforward: Fix the spins σ_{bi} for $i = 0, \dots, N/b - 1$, and integrate out the remaining ones. This will generate a renormalized Gibbs field with a Hamiltonian H' having the same form as the original Hamiltonian H, but containing a nearest-neighbor interaction J' and a magnetic field term m':

$$H' = -\left(J'\sum_{i=0}^{N/b-1}\sigma_{bi}\sigma_{b(i+1)} + m'\sum_{i=0}^{N/b-1}\sigma_{bi}\right). \tag{2}$$

The renormalized spin coefficients (J', m') and the original spin coefficients (J, m) are related by the decimation map:

$$\exp\left(C + J'\sigma_0\sigma_b + \frac{m'}{2}(\sigma_0+\sigma_b)\right) = \sum_{\sigma_1,\dots,\sigma_{b-1}}\exp\left(J\sum_{i=0}^{b-1}\sigma_i\sigma_{i+1} + \frac{m}{2}\sum_{i=0}^{b-1}(\sigma_i+\sigma_{i+1})\right), \tag{3}$$

where C is a normalization constant. Notice that to avoid double counting, we have assigned a "half" of the magnetic field m (m') to each spin.

We would like to obtain an explicit characterization of the renormalized model, but as the blocking factor b gets large, solving for (J', m') directly from (3) becomes very difficult. We thus take an alternative approach and investigate the decimation effect on the associated Markov chains. As there is no finite phase transition in one dimension, we follow the common practice and measure the nearest-neighbor interaction strength J (J') by the Boltzmann factor $k = e^{-2J}$ ($k' = e^{-2J'}$) instead. An explicit solution for (k', m') then follows from the Markov–Gibbs equivalence (Hammersley–Clifford theorem). The diagram below illustrates these ideas:

$$\begin{array}{ccc} \text{original Hamiltonian} & \overset{\text{(I)}}{\rightarrow} & \text{renormalized Hamiltonian} \\ \downarrow_{\text{(II)}} & & \uparrow_{\text{(IV)}} \\ \text{original Markov chain} & \overset{\text{(III)}}{\rightarrow} & \text{renormalized Markov chain} \end{array} \tag{4}$$

where:

- (II) and (IV) indicate the Markov–Gibbs equivalence (Hammersley–Clifford theorem).
- (I) is the decimation map on the Ising Hamiltonian (cf. (3)).
- (III) is the decimation map on the associated Markov chains (to be examined).

A key tenet of the renormalization group is its explanation of universality [13]. Thus we would also like to verify the widely-believed universality conjecture in this special case, which states that the linearization of the decimation transformation with blocking factor b about the two-dimensional fixed point ($k = m = 0$) has two real eigenvalues b^{y_T} and b^{y_H}, where $y_T = y_H = 1$. Suppose we start with a Hamiltonian that is close to critical. The decimation map will first drive it towards the fixed point for a large number of iterations, but eventually will drive it away. The singular behavior of the model arises from iterating the map infinitely many times, and the critical properties are determined by how much