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Market structure explained by pairwise interactions

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ABSTRACT

Financial markets are a typical example of complex systems where interactions between constituents lead to many remarkable features. Here we give empirical evidence, by making as few assumptions as possible, that the market microstructure capturing almost all of the available information in the data of stock markets does not involve higher order than pairwise interactions. We give an economic interpretation of this pairwise model. We show that it accurately recovers the empirical correlation coefficients; thus the collective behaviors are quantitatively described by models that capture the observed pairwise correlations but no higher-order interactions. Furthermore, we show that an order–disorder transition occurs, as predicted by the pairwise model. Last, we make the link with the graph-theoretic description of stock markets recovering the non-random and scale-free topology, shrinking length during crashes and meaningful clustering features, as expected.

1. Introduction

Complex systems are particularly interesting because they exhibit very sophisticated behaviors caused by, a priori, simple rules. Indeed, magnetic materials and neural networks, for instance, have some striking features such as phase transitions, memory, complicated equilibria structures and clustering. It is remarkable that these properties are caused by such simple interactions as pairwise ones. We believe that the markets are also driven by such simple rules and that the higher-order interactions encountered in financial systems are the pairwise ones. Typical characteristics of a complex system are numerous entities and interaction rules (with a degree of non-linearity), all leading to the emergence of collective behaviors. Those behaviors in general depend more on the interactions (e.g., their scaling and their order) and their effects than on the intrinsic nature of the elementary constitutive entities taken individually. The market can be viewed as such a system. The entities can be stocks or traders interacting through non-obvious rules. We note that we should interpret *interaction* at the larger sense of mutual or reciprocal influence.

What one knows is that the markets exhibit features such as synchronization [1], structural reorganization [2,3], power laws [4,5], hierarchical and non-randomness [6]. What one does not know is the true market dynamics. Even if trading rules are known, microscopic equations of motion are not known. This is a fundamental difference between finance and physics (or neuroscience).

A natural approach, given the above considerations, is a statistical modeling collecting and using at best the available amount of information and allowing (in a certain sense) the emergence of critical properties. This is exactly the purpose of the maximum entropy modeling in complex systems theory. Indeed the maximum entropy principle (MEP) allows the selection of the less restricting model on the basis of incomplete information [7]. We choose this data-based approach to avoid the use of any particular microscopic schemes (e.g. trader–agent-based rules, a priori unknown) which are difficult to assess experimentally or to avoid any analogy (even if some of such models are valuable [8]). The reason is that, even if one does not know the underlying microscopic processes, the macroscopic collective behaviors can still be described by an *effective*

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model. One has long experience of this powerful approach in the description of phase transitions and magnetic materials [9]. More recently, it has led to valuable results about the description of real neural networks [10]. Moreover, this approach also has counterparts in economics. Indeed, in addition of the statistical meaning of entropy, one can interpret it as a measure of economic activity [11] and it is linked to the central concept of *utility* of many interacting economic entities [12,13].

An important outcome of such a modeling is a convenient simplified version of the real interaction structure that is still consistent with the data. In the following, we derive the model in this point of view and we study the structural properties of the resulting complex network. The critical properties will be investigated in another work.

The paper is organized as follow. In Section 2, we present the model, its economic interpretation and the link between the interaction matrix and the moments. In Section 3, we give evidence that the information embedded in the data is mostly explained by the pairwise but no higher-order interactions. In Section 4, we show an order–disorder transition through actual data. In Section 5, we highlight the properties of the interaction matrix and its link to the crises. Finally, in Section 6, we explain the link with the graph-theoretic approach and the topological evolution of the market network.

2. The model

2.1. Model derivation

The aim is to set up a statistical model describing the market state. This requires a way to infer the probability distribution in order to get the observables (here, the associated moments). The model will also allow the study of the market structure. All these quantities will be defined below. We consider a set of N market indices or N stocks with binary states s_i ($s_i = \pm 1$ for all $i = 1, \ldots, N$). A system configuration will be described by a vector $\mathbf{s} = (s_1, \ldots, s_N)$. The binary variable will be equal to 1 if the associated index is bullish and equal to -1 if not. A configuration $(s_1, \ldots, s_N)$ is a binary version of the index returns. One knows that this approximation is already useful in the description of neural populations [10] and that neural networks are similar to financial networks [6]. We may think that it will also be the case in finance; this will be justified a posteriori as the model gives consistent results.

We seek to establish a less structured model explaining only the measured index mean orientation $q_i = \langle s_i \rangle$ and instantaneous pairwise correlations $q_{kl} = \langle s_k s_l \rangle$. The brackets $\langle \cdot \rangle$ denote the average with respect to the unknown distribution $p(\mathbf{s})$. As the entropy of a distribution measures the randomness or the lack of interaction among the binary variables, a way to infer such probability distribution knowing the mean orientations and the correlations is the maximum entropy principle. Jaynes showed how to derive the probability distribution using the maximum entropy principle [14]; for supplementary information see Ref. [7]. It consists in the following constrained maximization:

$$\begin{aligned} \max \quad & S(\mathbf{s}) = -\sum_{\{\mathbf{s}\}} p(\mathbf{s}) \, \log p(\mathbf{s}) \\ \text{s.t.} \quad & \sum_{\{\mathbf{s}\}} p(\mathbf{s}) = 1, \qquad \sum_{\{\mathbf{s}\}} p(\mathbf{s}) s_i = q_i, \quad \sum_{\{\mathbf{s}\}} p(\mathbf{s}) s_i s_j = q_{ij}. \end{aligned} \tag{1}$$

The resulting two-agent distribution $p_2(\mathbf{s})$ is the following:

$$p_2(\mathbf{s}) = \mathcal{Z}^{-1} \exp\left(\frac{1}{2} \sum_{i,j}^{N} J_{ij} s_i s_j + \sum_{i=1}^{N} h_i s_i \right) \equiv \frac{e^{-\mathcal{H}(\mathbf{s})}}{\mathcal{Z}}, \tag{2}$$

where J_{ij} and h_i are Lagrange multipliers and $\mathcal{Z}$ a normalizing constant (the partition function). They can be expressed in terms of partial derivatives of the entropy as

$$\frac{\partial S(\mathbf{s})}{\partial q_i} = -h_i \qquad \frac{\partial S(\mathbf{s})}{\partial q_{ij}} = -J_{ij}. \tag{3}$$

Thus preferences are conjugated to mean orientations and interaction strengths to pairwise correlations. Including higher-order correlations in constraints in (1) could bring more information and thus decrease the maximum entropy. We will show below that this will not be the case.

The Gibbs distribution (2) is similar to the one given by Brock and Durlauf in the discrete choice problem [12] and to the one in stochastic models in macroeconomics [11], and also to the Ising model used in description of magnetic materials and neural networks [9,10]. It is also a special case of Markov random fields [15]. It is to be noted that the Gibbs distribution and Shannon entropy naturally arise from the stochastic modeling in economics; this is discussed in Ref. [11].

We obtain the parameters $\{J_{ij}, h_i\}$ by performing explicitly the maximization (1) so that the theoretical moments $\langle s_i \rangle$ and $\langle s_i s_j \rangle$ match the measured ones q_i and q_{ij}. We note that this requires the computation of 2^N terms. If this number is large, the computation will take a while, and we can benefit from one of the methods described in Ref. [16].

Last, we show how the cumulants are obtained from this model and their relation to interaction strengths. As the statistical model (2) is expressed as a Gibbs distribution, we have the relations

$$\langle s_{i_1} \cdots s_{i_N} \rangle_c = \partial^N \ln \mathcal{Z} / \partial h_{i_1} \cdots \partial h_{i_N}, \tag{4}$$

where $\langle \cdot \rangle_c$ is the cumulant average [17]; it gives the relation between $\mathbf{J}$ and the correlation functions. If the partition function $\mathcal{Z}$ cannot be explicitly computed, we can use the Plefka series [18] or a variational cumulant expansion [19].