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Critical properties of contact process on the Apollonian network

L.F. da Silva [a], R.N. Costa Filho [a], D.J.B. Soares [b], A. Macedo-Filho [c], U.L. Fulco [c,*], E.L. Albuquerque [c]

[a] Departamento de Física, Universidade Federal do Ceará, 60451-970, Fortaleza-CE, Brazil

[b] Departamento de Física, Universidade Federal de Campina Grande, CES Campus Cuité, 58175-000, Cuité-PB, Brazil

[c] Departamento de Biofísica e Farmacologia, Universidade Federal do Rio Grande do Norte, 59072-970, Natal-RN, Brazil

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ABSTRACT

We investigate an epidemic spreading process by means of a computational simulation on the Apollonian network, which is simultaneously small-world, scale-free, Euclidean, space-filling and matching graphs. An analysis of the critical behavior of the Contact Process (CP) is presented using a Monte Carlo method. Our model shows a competition between healthy and infected individuals in a given biological or technological system, leading to a continuous phase transition between the active and inactive states, whose critical exponents $\beta/\nu_\perp$ and $1/\nu_\perp$ are calculated. Employing a finite-size scaling analysis, we show that the continuous phase transition belongs to the mean-field directed percolation universality class in regular lattices.

1. Introduction

In the last few years, important steps toward a better understanding of critical phenomena in complex networks have been made [1,2]. In particular, the research community has devoted a great deal of attention to the study of the dynamical processes on these networks [3–5], which can have important implications in the study of real processes such as social systems [6], virus spreading in computers, and traffic in technological information systems [7,8], as well as the spread of epidemic diseases [9,10]. For the latter process, the contamination of vertices on the complex network, the existence of non-equilibrium phase transitions, and the identification of the type of transition, are the tools to study the spreading of epidemic processes. From the theoretical point of view, despite its simplicity, the mean-field (MF) model is the usual technique used to study such networks' behavior [11,12], since it describes qualitatively well most of the phase transitions, particularly the critical behavior of complex networks which belong to the MF universality class [13].

Complex networks describe many systems in nature and society [14,15], and most of them share three features: power-law degree distribution, small average path length, and high clustering coefficient. Regarding their topology, they are usually divided into three large classes [16]: random networks, where all the nodes are randomly connected [17], scale-free network, presenting connected graph with the property that the number of links originating from a given node exhibits a power law distribution [18], and the small-world one, showing structures where the diameter or the average shortest path ℓ increases logarithmically with the system size N (number of nodes) [19].

Among the complex networks, the Apollonian network is a special one since it belongs to a particular class of deterministic networks that are scale-free, display small-world effect, can be embedded in a Euclidean lattice, and show space-filling

* Corresponding author. Tel.: +55 84 32153793; fax: +55 84 32153791.
E-mail address: umbertofulco@gmail.com (U.L. Fulco).

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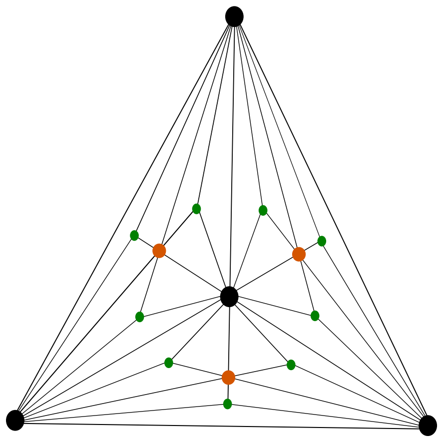

Fig. 1. (colour online) Building process of the Apollonian networks.

as well as matching graph properties [20,21]. Previous studies have already been made about their topological features, and their effect on the behavior of a variety of transport and growth models [22–27]. In particular, the nature of the one-electron eigenstates (energy spectrum) of a free-electron gas in the Apollonian network has been recently investigated presenting several unique features such as delta-like singularities, gaps and minibands, as well as localized, extended and critical electronic states [28].

In this work we focus our attention on the properties of the Contact Process (CP) model in the Apollonian network. The Contact Process model was introduced by Harris in 1974 [29] and is one of the most investigated epidemic spreading models. According to the MF model, the critical behavior of the system depends on the distribution degree γ. More precisely, there are three regions; for $\gamma > 3$ the critical exponents are $\beta = 1$ and $\nu_\perp = 2$; for $2 < \gamma < 3$, we have $\beta = 1/(\gamma - 2)$ and $\nu_\perp = (\gamma - 1)/(\gamma - 2)$; and for $\gamma < 2$ there is no transition for a finite λ [30,3,13,31]. A variation of the CP model, the so-called susceptible–infected–susceptible (SIS) model is also widely applied in complex networks [32,33]. The CP model is defined as a d-dimensional lattice L^d, where each site is identified as individuals in two states: healthy (inactive) and infected (active). The dynamics of the interaction between the individuals obeys local Markovian rules. In this context, healthy individuals become infected at a rate that depends on the number of infected neighbors, while the infected one becomes healthy at a rate λ, with the density of infected individuals ρ_D going to zero for high values of λ. On the other hand, it evolves to a stationary state with a finite density of infected individuals for low values of λ, with a critical λ_c separating the active and inactive regimes. This model, therefore, exhibits a continuous phase transition between a stationary active state and one absorbing state ($\rho_D = 0$). Numerical simulations have shown that this transition between active and inactive regimes belongs to the directed percolation universality class [34].

In this work, we perform an extensive numerical simulation of the contact process on the Apollonian network, aiming to verify the type of phase transition and to estimate the critical properties of the network. We also verified how the topology of this network affects the critical properties, looking for which universality class this system belongs to.

This paper is organized as follows. In Section 2 we present the Apollonian network model according to the current literature. In Section 3 we consider the numerical simulations of the CP model, as well as the discussion of the numerical results. Section 4 is devoted to some concluding remarks.

2. Apollonian networks

The Apollonian network is a deterministic version of the problem of space-filling packing of spheres. In the two-dimensional version, we consider the problem of a space-filling packing of disks according to the ancient Greek mathematician Apollonius of Perga Ref. [35]. Three disks touch each other and the circle between them is filled by another disk that touches all the previous three, forming much smaller circles that are then filled again and so on (see Fig. 1). Connecting the centers of the touching disks by lines, one obtains a network which gives a triangulation that physically corresponds to the force network of a dense granular packing. This network resembles the graphs introduced by Dodds Ref. [36] for the case of random packings, and has also been used in the context of porous media [37].

The above described Apollonian network has the following properties:

(i) It is *scale-free*. The cumulative of the degree distribution, $P(k) = \sum_{k' \geq k} m(k', n)/N_n$, exhibits a power-law distribution $P(k) \propto k^{1-\gamma}$ with the exponent $\gamma = 1 + (\ln)(3)/(\ln)(2) \approx 2.585$ characterizing the stationary distribution $P(k)$. Here k is the vertex degree (connectivity), and $N_n = 3 + (3^{n+1} - 1)/2$ is the number of vertices at each generation n;

(ii) It displays *small-world* effect. The average length of the shortest path ℓ, between two vertices, grows slower than any positive power of the system size N. In fact, the Apollonian network behaves like a random graph, i.e., $\ell \propto [\ln(N)]^{3/4}$, and has a clustering coefficient $C = 0.828$ in the limit of large N. For instance, the clustering coefficient for the collaborations