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Minireview

Multifractal models via products of geometric OU-processes: Review and applications

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ABSTRACT

This paper reviews a class of multifractal models obtained via products of exponential Ornstein–Uhlenbeck processes driven by Lévy motion. Given a self-decomposable distribution, conditions for constructing multifractal scenarios and general formulas for their Renyi functions are provided. Together with several examples, a model with multifractal activity time is discussed and an application to exchange data is presented.

Contents

1. Introduction

Since Mandelbrot (in particular Refs. [1–3]) developed and popularized the concept of fractals and multifractals, and advocated their use in the explanation of observed features of time series arising in natural sciences, there has been ongoing interest by researchers in a variety of disciplines in widening their application.

Models with multifractal scaling have been used in many applications in hydrodynamic turbulence, genomics, computer network traffic, commodity prices, financial markets etc., an arbitrary selection of papers and the references therein to which the interested reader is referred are those of Refs. [4–9,3,10–16].

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In this paper we are going to discuss a class of multifractal models originally introduced by Anh et al. [17] and show they provide a useful and flexible family of models for applications. We will do so by analyzing some foreign exchange rates series. Evidence of multifractal feaures in this area is well documented, see, for example, Refs. [18–21] and the references therein. The aim here is not that of discovering again multifractality in the data, but rather to discuss and present a variety of models with multifractality which can be easily adapted to the most varied settings. As we will see, one of the relevant practical features of these models is the easy derivation of scaling properties of the model from basic quantities of the underlying processes.

We will give a short description of the main features of fractals and multifractals in the next section. Section 3 discusses the construction of a multifractal process based on the products of geometric Ornstein–Uhlenbeck (OU) processes and provides several examples. Section 4 discusses the fitting of the models to real financial data.

2. Multifractals

There are two main models for fractals that occur in nature. Generally speaking, fractals are either statistically self-similar or they are multifractals.

The definition of a multifractal is motivated by that of a stochastic process X_t which satisfies a relationship of the form

$$\{X(ct)\} \stackrel{d}{=} \{M(c)X(t)\}, \quad t \geq 0 \tag{2.1}$$

for positive $0 < c < 1$, where M is a random variable independent of X and equality is in finite-dimensional distributions. In the special case $M(c) = c^H$, the multifractal reduces to a self-similar fractal where the parameter $0 < H < 1$ is known as the Hurst parameter, named after the British engineer Harold Hurst (whose work on Nile river data played an important role in the development of self-similar processes). For a more detailed review of self-similar processes, see Ref. [22]. The actual definition of a multifractal process, as given in Ref. [3], is defined in terms of the moments of the process and includes processes satisfying (2.1).

A stochastic process $X(t)$ is multifractal if it has stationary increments and there exist functions $c(q)$ and $\tau(q)$ and positive constants Q and T such that $\forall q \epsilon Q = [q_-, q_+]$, $\forall t \epsilon [0, T]$,

$$E(|X(t)|^q) = c(q)t^{\tau(q)+1}, \tag{2.2}$$

where $\tau(q)$ and $c(q)$ are both deterministic functions of q. $\tau(q)$ is called the scaling function and takes into account the influence of the time t on the moments q, and $c(q)$ is called the prefactor. While this definition is the standard definition of a multifractal process, most processes studied as multifractals only obey it for particular values of t, or sometimes for asymptotically small t. The condition of stationary increments is also quite often relaxed. Conversely, Taqqu et al. [23] tests the scaling properties of the increments of $X(t)$ instead of the process itself. If this method is used then the subtraction of the mean $E(X(t + 1) - X(t))$ to $X(t + 1) - X(t)$ may be required to ensure a fair investigation, because such a stationary process cannot be self-similar or even asymptotically self-similar if it has non-zero mean. For our case, we will find that $E(X(t + 1) - X(t)) = 0$ for each of our data sets. It follows from (2.2) that

$$\log E(|X(t)|^q) = \log c(q) + (\tau(q) + 1) \log t \tag{2.3}$$

and so $X(t)$ is multifractal if for each $q \in Q$, $\log E|X(t)|^q$ scales linearly with $\log t$ and the slope is $\tau(q) + 1$. To explain the notion of the scaling function $\tau(q)$, consider the particular case of the fractional Brownian motion, a self-similar process. A fractional Brownian motion, with a Hurst exponent H, satisfies $X(t) \stackrel{d}{=} t^H X(1)$, which implies that $E(|X(t)|^q) \stackrel{d}{=} t^{Hq} E(|X(1)|^q)$. Here we obtain the prefactor $c(q) = E(|X(1)|^q)$, and the scaling function $\tau(q) = Hq - 1$. So the scaling function is linear if the process is self-similar. Alternatively, the process is multifractal if it has multiscaling properties that imply nonlinearity of the scaling function. Mandelbrot et al. [3] showed that the scaling function is concave for all multifractals with the following argument. Let ω_1, ω_2 be positive weights with $\omega_1 + \omega_2 = 1$ and let $0 \leq q_1, q_2 \leq q_+$ and $q = q_1\omega_1 + q_2\omega_2$. Then by Hölder inequality

$$E|X(t)|^q \leq (E|X(t)|^{q_1})^{\omega_1} (E|X(t)|^{q_2})^{\omega_2} \tag{2.4}$$

and so

$$\log c(q) + \tau(q) \log t \leq (\omega_1 \tau(q_1) + \omega_2 \tau(q_2)) \log t + (\omega_1 \log c(q_1) + \omega_2 \log c(q_2)). \tag{2.5}$$

Letting t go to zero we have $\tau(q) \geq \omega_1\tau(q_1) + \omega_2\tau(q_2)$ and so τ is concave. If $T = \infty$ we can let t go to ∞ and we get the reverse inequality $\tau(q) \leq \omega_1\tau(q_1) + \omega_2\tau(q_2)$. It follows that $T = \infty$ implies that τ is linear and so $X(t)$ is self-similar. An important associated concept is the multifractal spectrum. It is the Legendre transform of the scaling function $\tau(q)$ and is given by

$$f(\alpha) = \inf_q [q\alpha - \tau(q)], \tag{2.6}$$

where it is defined. For self-similar processes it is only defined at H with $f(H) = 1$. The multifractal spectrum plays an important role in multifractal measures, where it represents the fractal dimensions of sets where the measure has certain limiting intensities. The analogous definition for multifractal processes is the dimension of sets with local Hölder exponent α (see Ref. [24] for details).