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Unified physics of stretched exponential relaxation and Weibull fracture statistics

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ABSTRACT

The complicated nature of materials often necessitates a statistical approach to understanding and predicting their underlying physics. One such example is the empirical Weibull distribution used to describe the fracture statistics of brittle materials such as glass and ceramics. The Weibull distribution adopts the same mathematical form as proposed by Kohlrausch for stretched exponential relaxation. Although it was also originally proposed as a strictly empirical expression, stretched exponential decay has more recently been derived from the Phillips diffusion-trap model, which links the dimensionless stretching exponent to the topology of excitations in a glassy network. In this paper we propose an analogous explanation as a physical basis for the Weibull distribution, with an ensemble of flaws in the brittle material serving as a substitute for the traps in the Phillips model. One key difference between stretched exponential relaxation and Weibull fracture statistics is the effective dimensionality of the system. We argue that the stochastic description of the flaw space in the Weibull distribution results in a negative dimensionality, which explains the difference in magnitude of the dimensionless Weibull modulus compared to the stretching relaxation exponent.

1. Introduction

Materials science encompasses a broad range of chemistry and physics. The complicated multi-body interactions in condensed matter systems often require a statistical approach for predicting and understanding their physical properties. Even deterministic processes such as brittle fracture are often best represented using a probabilistic interpretation, since the particular distribution of flaws in a given sample is generally stochastic in nature. Assuming a single type of dominant flaw population, the probability of fracture of a brittle material such as glass is well described by the empirical Weibull distribution [1]:

$$F(\sigma) = 1 - \exp\left[-\frac{(\sigma - \sigma_{\min})^m}{\sigma_0}\right], \tag{1}$$

where $F(\sigma)$ is the probability of failure under a tensile stress σ, σ_0 is the characteristic strength, $\sigma_{\min}$ is the minimum strength (below which no failure will occur), and the dimensionless exponent m is known as the Weibull modulus. The Weibull distribution was proposed without any theoretical basis for its economy of parameters and has met with remarkable success in describing the failure statistics of brittle materials such as glass and ceramics [2].

Interestingly, the functional form adopted by Weibull for his distribution is identical to that proposed more than a century prior by Kohlrausch to describe nonexponential relaxation behavior [3]. As a nonequilibrium material, a glass

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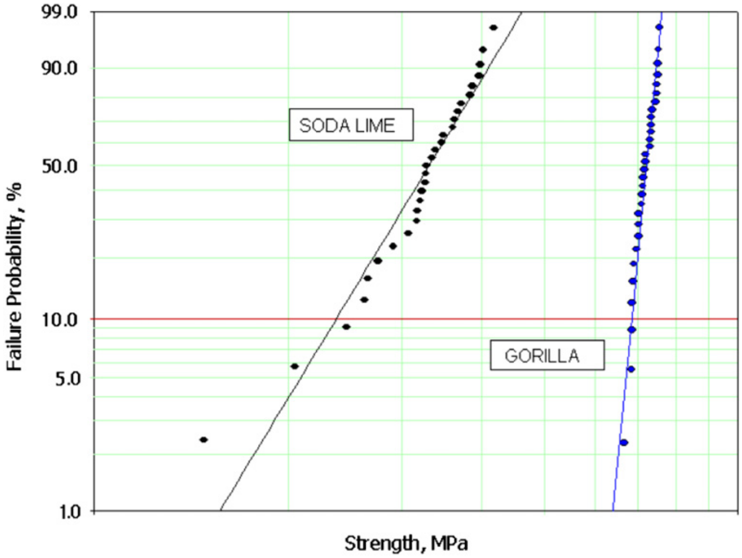

Fig. 1. Failure probability as a function of applied stress for soda lime silicate versus Corning Gorilla® glass [8]. The slopes of the straight-line fits correspond to the Weibull moduli for these glasses, where a steeper slope (i.e., greater Weibull modulus) is indicative of less variability in the strength of the glass.

is continuously relaxing toward its equilibrium supercooled liquid state [4–6], which is captured by the nonexponential relaxation function [3]:

$$f(t) = \exp\left[-\left(\frac{t}{\tau}\right)^{\beta}\right]. \tag{2}$$

This is the so-called "stretched exponential" relaxation function, $f(t)$, which decays from an initial value of $f(0) = 1$ to $f(\infty) = 0$ in the limit of long time t. The parameter τ is the characteristic relaxation time for the decay, and the exponent β in this case is known as the "stretching exponent". As with the Weibull distribution, the Kohlrausch function was proposed empirically for its economy of parameters and notable success in fitting experimental data [7].

Of particular interest in both Weibull fracture statistics and stretched exponential relaxation is the value of the dimensionless exponent, m or β, respectively. In the former case, a higher value of the Weibull modulus m is generally desirable since it is indicative of less spread in the observed strength of a material (see Fig. 1, for example). A larger value of m is indicative of greater predictability and hence greater reliability. In the ideal case of a perfectly uniform flaw population, the Weibull modulus would approach its theoretical upper limit of $m \to \infty$. In practice, typical values of m are on the order of 5–50 [2].

In the latter case of stretched exponential relaxation, the dimensionless stretching exponent satisfies $0 < \beta \leq 1$, where the upper limit of unity corresponds to simple exponential relaxation. While traditionally left as an empirical fitting parameter, the diffusion-trap theory of Phillips [7] has shown that the value of β for homogeneous glassy systems can be derived from purely topological considerations. Based on these topological arguments, the Phillips theory predicts certain universal values of β, most notably $\beta = 3/5$ for systems dominated by short-range forces and $\beta = 3/7$ for those dominated by long-range relaxation pathways. This bifurcation of the stretching exponent has recently been confirmed in a decisive experiment by Potuzak et al. [9]

In this paper, we consider the common physics underlying both Weibull fracture statistics and stretched exponential relaxation. We propose that Weibull fracture statistics can be viewed in a fashion analogous to stretched exponential relaxation according to the Phillips diffusion-trap model. In this view, the Weibull distribution can be interpreted as an analog to stretched exponential relaxation in an ensemble-strength space. These common underlying physics point to the topological nature of fracture statistics, where the ideal upper limit of the Weibull modulus ($m \to \infty$) is obtained for a negative dimensionality of the iso-stress space equal to $d = -2$. We argue that the negative dimensionality of the iso-stress space is a direct result of the stochastic nature of the flaw distribution, with the fracture of each sample being governed by its single most severe flaw.

2. Stretched exponential relaxation

A comprehensive physical understanding of the relaxation behavior of glass is of vital importance for high-tech applications, including optical fiber [10] and substrate glass for liquid crystal displays [11–13]. Relaxation behavior is also an important consideration for chemically strengthened cover glass for personal electronic devices [14,15]. Glass relaxation is