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A connection between a system of random walks and rumor transmission

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HIGHLIGHTS

- We show a connection between rumor transmission and random walks.
- We present a coupling between the Maki–Thompson epidemic model and the frog model.
- We discuss applications of the frog model which are relevant to biological dynamics.

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ABSTRACT

We establish a relationship between the phenomenon of rumor transmission on a population and a probabilistic model of interacting particles on the complete graph. More precisely, we consider variations of the Maki–Thompson epidemic model and the "frog model" of random walks, which were introduced in the scientific literature independently and in different contexts. We analyze the Markov chains which describe these models, and show a coupling between them. Our connection shows how the propagation of a rumor in a closed homogeneously mixing population can be described by a system of random walks on the complete graph. Additionally, we discuss further applications of the random walk model which are relevant to the modeling of different biological dynamics.

1. Introduction

The strong and fruitful interplay between probability theory and physics is well known. Many classical probability models were inspired from physics, and were developed in order to answer questions of physical and mathematical interest. Often the answers give rise to new probability models, for which new questions appear, and so on. The beauty of this interplay is particularly illustrative when the new theoretical models can be used to describe a phenomenon different from the original one. Interesting examples of such evolution can be found in the theories of branching processes, percolation, and random walks, among others. The purpose of this paper is to explore this type of connection, by showing the strong relation between a model of interacting random walks existing in the probabilistic literature and the phenomenon of rumor transmission on a population. For this, we discuss the equivalence between two mathematical models which, while being proposed originally in different contexts, are in fact modeling the same phenomenon.

The model of random walks we consider is a variation of the system called the "frog model" amongst probabilists. The basic version of this model is defined on an infinite connected graph. At time zero, there is a particle at each site of the graph,

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and each particle can be in one of two states: active or inactive. An active particle performs a discrete-time random walk on the graph. When it hits an inactive particle, that inactive particle becomes active, and starts an independent random walk on the graph. Additionally, before each step, each active particle survives or dies with respective probabilities p and $(1-p)$. The main motivation given for this stochastic process is that it models information diffusion in a population. One can think of an active particle as an agent carrying an item of information, which is shared with all the inactive particles (ignorant agents) it meets on its way. This interpretation, however, has not been much explored in the literature.

The first subjects of interest for the frog model were phase transitions with respect to the survival probability of the process and asymptotic values for critical parameters in $\mathbb{Z}^d$ and regular trees; see Ref. [1]. In Ref. [2], a shape theorem for the growing set of sites visited by active particles is proved. More recently, the critical probability was studied on homogeneous trees [3], and some variations were proposed in order to understand the phenomena of extinction and survival of processes with distinct dynamics on $\mathbb{Z}$ [4]. Regarding the behavior of the frog model and its variations on finite graphs, as far as we know, the main object under analysis is the limiting proportion of vertices visited by active particles on the complete graph of N vertices, as N goes to ∞ [5–7]. The interesting aspect we address here is that, while this model was developed in order to answer questions of mathematical interest, it has reached a point at which it describes very well how a rumor spreads out in a population.

To start presenting this connection, we describe the rumor spreading model introduced by Maki and Thompson [8] as a basic version of the model of Daley and Kendall [9], which has attracted great interest in recent years. In this model, which we abbreviate to the [MT] model, a closed homogeneously mixing population of $N+1$ individuals is subdivided into three classes: those not aware of the rumor, those who are spreading it, and those who know the rumor but have ceased communicating it. These classes are called *ignorants*, *spreaders*, and *stiflers*, respectively. Once a rumor appears in a population, it is propagated by directed contacts between spreaders and other individuals, according to the following set of rules.

1. When a spreader interacts with an ignorant, the ignorant becomes a spreader.
2. Whenever a spreader contacts a stifler, the spreader turns into a stifler.
3. When a spreader meets another spreader, the initiating one becomes a stifler and the other continues spreading.

In the last two cases, it is said that the spreader was involved in a *stifling experience* [10]. Notice that the rumor process must eventually finish, namely, when there are no more spreaders in the population, so it is of interest to study the proportion of remaining ignorants. The first rigorous result for the [MT] model, proved by Sudbury [11], establishes that, for a process initiating with one spreader, the ultimate proportion of ignorants converges in probability to 0.203 as N tends to ∞. This means that, for a large population size, the rumor spreads through the population, but, with high probability, approximately a fifth of the people are not aware of it at the moment that the last spreader disappears. Sudbury's result was later generalized by Watson [12], who proved that the distribution of the proportion of the population never hearing the rumor is asymptotically normal, with mean 0.2032 and variance $0.2728\,N^{-1}$. An analysis of this proportion for the deterministic version of the model starting from a general initial condition was presented by Belen and Pearce [13]. Recently, some variations have been considered in order to study whether the rumor survives or not on regular and complex networks (see for instance [14–18], and the references therein). We refer to Daley and Gani [10, Chap. 5] for an excellent account on the subject of rumor models.

The rest of this paper is organized as follows. First, we present the formal descriptions of the rumor model and the random walk system that we want to compare. Parts of these descriptions have already appeared in Refs. [6,19], but we include them for the sake of completeness. Next, we present the relation between these models, which shows the process of propagation of a rumor through a random walk system. We end with a conclusion and a discussion of possible applications.

2. The random stifling Maki–Thompson model

We consider the generalization of the [MT] model called the *random stifling* [MT] model, which was formulated in Ref. [19]. The main idea is that each spreader decides to stop propagating the rumor right after being involved in a random number of stifling experiences.

To fix the notation, let us give a formal description of the basic [MT] model. We consider a closed homogeneously mixing population of size $N+1$, and denote the number of ignorants, spreaders, and stiflers at time t by $X(t)$, $Y(t)$, and $Z(t)$, respectively. Initially, $X(0)=N$, $Y(0)=1$, $Z(0)=0$, and $X(t)+Y(t)+Z(t)=N+1$ for all t. Having in mind the rules described in the introduction, the process $\{(X(t),Y(t))\}_{t\geq 0}$ is a continuous-time Markov chain with increments and corresponding rates given by

increment	rate
$(-1, 1)$	XY,
$(0, -1)$	$Y(N-X)$.

This means that the transition probabilities of the process satisfy the equations $P((X(t+h),Y(t+h))=(X-1,Y+1)\mid(X(t),Y(t))=(X,Y))=X\,Y\,h+o(h)$ and $P((X(t+h),Y(t+h))=(X,Y-1)\mid(X(t),Y(t))=(X,Y))=Y(N-X)\,h+o(h)$. The first transition corresponds to a spreader telling the rumor to an ignorant, who becomes a spreader. The second transition corresponds to a spreader meeting another spreader or a stifler, in which case the spreader loses interest in propagating the rumor and becomes a stifler.

As already explained, in the random stifling [MT] model, an ignorant individual is allowed to have a random number of stifling experiences once he/she is told the rumor. More precisely, let R be a nonnegative integer-valued random variable