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Space–time fractional diffusion equations and asymptotic behaviors of a coupled continuous time random walk model

Long Shi [a,b], Zuguo Yu [a,*], Zhi Mao [a], Aiguo Xiao [a], Hailan Huang [a]

[a] *School of Mathematics and Computational Science, Xiangtan University, Xiangtan, Hunan 411105, China*

[b] *Institute of Mathematics and Physics, Central South University of Forest and Technology, Changsha, Hunan 410004, China*

HIGHLIGHTS

- A type of coupled continuous time random walk model is considered.
- The asymptotic behaviors of the coupled jump PDF are discussed.
- The corresponding fractional diffusion equations are derived.
- The asymptotic behaviors of the PDF of the waiting time are discussed.
- The asymptotic behaviors of the conditional PDF of the jump length are discussed.

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ABSTRACT

In this paper, we consider a type of continuous time random walk model where the jump length is correlated with the waiting time. The asymptotic behaviors of the coupled jump probability density function in the Fourier–Laplace domain are discussed. The corresponding fractional diffusion equations are derived from the given asymptotic behaviors. Corresponding to the asymptotic behaviors of the joint probability density function in the Fourier–Laplace space, the asymptotic behaviors of the waiting time probability density and the conditional probability density for jump length are also discussed.

1. Introduction

The continuous time random walk (CTRW) theory, which was introduced in the 1960s by Montroll and Weiss to describe a walker hopping randomly on a periodic lattice with the steps occurring at random time intervals [1], has been applied successfully in many fields (see, e.g., the reviews [2–4] and references therein).

In a continuum one-dimensional space, the CTRW scheme is characterized by a jump probability density function (PDF) $\psi(x, t)$, which is the probability density that the walker makes a jump of length x after some waiting time t. Let $P(x, t)$ be the PDF of finding the walker at a given place x and at time t with the initial condition $P(x, 0) = \delta(x)$. A CTRW process can be described by the following integral equation [3]:

$$P(x,t) = \int_{-\infty}^{+\infty} \mathrm{d}x' \int_0^t \psi(x-x', t-t') P(x', t') \mathrm{d}t' + \delta(x)\Phi(t), \tag{1}$$

* Corresponding author. Tel.: +86 731 58298760.
E-mail addresses: slong8008@163.com (L. Shi), yuzg1970@yahoo.com, yuzuguo@aliyun.com (Z. Yu).

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where $\Phi(t) = 1 - \int_0^t \varphi(\tau)\mathrm{d}\tau$ is the probability of not having made a jump until time t and $\varphi(t) = \int_{-\infty}^{+\infty} \psi(x,t)\mathrm{d}x$ is the waiting time PDF.

Fractional diffusion equations (FDEs) arise quite naturally as the limiting dynamic equations of CTRW models with temporal and/or space memories [5]. The asymptotic relation between the CTRW models and the fractional diffusion processes was studied firstly by Balakrishnan in 1985, dealing with the anomalous diffusion in one dimension [6]. Later, many authors discussed the relation between CTRW and FDEs [3–5,7–19]. However, the usual assumption in most of these works is that the CTRW is decoupled, which means that the jump lengths and the waiting times are independent. Recently, coupled CTRW models have attracted more attention [20–25]. Here, we focus on coupled CTRW models with the jump length correlated with the waiting time [25], i.e. $\psi(x,t) = \varphi(t)\lambda(x|t)$, and derive the corresponding FDEs from the asymptotic behaviors of the waiting time PDF $\varphi(t)$ and the jump PDF $\psi(x,t)$ in the Fourier–Laplace space.

This paper is organized as follows. In Section 2, we introduce a space–time fractional diffusion equation which can be obtained from the standard diffusion equation by replacing the first-order time derivative and/or the second-order space derivative by a Caputo derivative of order $\alpha \in (0,2]$ and/or a Riesz derivative of order $\beta \in (0,2]$, respectively. In Section 3, the asymptotic behaviors of the jump PDF $\psi(x,t)$ in the Fourier–Laplace domain are given and the corresponding FDEs are derived. In Section 4, corresponding to the asymptotic behaviors of the jump PDF $\psi(x,t)$ in the Fourier–Laplace domain, the asymptotic behaviors of the waiting time PDF $\varphi(t)$ and the conditional PDF of the jump length $\lambda(x|t)$ are discussed. In Section 5, some conclusions are presented.

2. The space–time fractional diffusion equation

We consider a space–time FDE [10]

$$ {}_0^C D_t^\alpha u(x,t) = K \frac{\partial^\beta u(x,t)}{\partial |x|^\beta}, \quad x \in R,\ t > 0, \tag{2} $$

where $u(x,t)$ is the field variable, K is the generalized diffusion constant and the real parameters α, β are restricted to the range $0 < \alpha \le 2$, $0 < \beta \le 2$.

In Eq. (2), the time derivative is the Caputo fractional derivative of order α, defined as [26]

$$ {}_0^C D_t^\alpha g(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t \frac{g^{(n)}(\tau)\mathrm{d}\tau}{(t-\tau)^{\alpha+1-n}}, & n-1 < \alpha < n, \\ g^{(n)}(t), & \alpha = n \in N, \end{cases} \tag{3} $$

and the space derivative is the Riesz fractional derivative of order β, defined as [27]

$$ \frac{\mathrm{d}^\beta}{\mathrm{d}|x|^\beta} f(x) = \begin{cases} \Gamma(1+\beta) \dfrac{\sin(\beta\pi/2)}{\pi} \displaystyle\int_0^{+\infty} \frac{f(x+\xi) - 2f(x) + f(x-\xi)}{\xi^{1+\beta}} \mathrm{d}\xi, & 0 < \beta < 2, \\ \dfrac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}, & \beta = 2. \end{cases} \tag{4} $$

Let

$$ \widehat{f}(k) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{+\infty} f(x) \mathrm{e}^{\mathrm{i}kx} \mathrm{d}x \tag{5} $$

be the Fourier transform of $f(x)$ and

$$ \widetilde{g}(s) = \mathcal{L}\{g(t)\} = \int_0^{+\infty} g(t) \mathrm{e}^{-st} \mathrm{d}t \tag{6} $$

be the Laplace transform of $g(t)$.

Now, let us recall the following fundamental formulas about the Laplace transform of the Caputo fractional derivative of order α and the Fourier transform of the Riesz fractional derivative of order β:

$$ \mathcal{L}\{{}_0^C D_t^\alpha g(t)\} = s^\alpha \widetilde{g}(s) - \sum_{m=0}^{n-1} s^{\alpha-1-m} g^{(m)}(0), \quad n-1 < \alpha \le n, \tag{7} $$

$$ \mathcal{F}\left\{ \frac{\mathrm{d}^\beta}{\mathrm{d}|x|^\beta} f(x) \right\} = -|k|^\beta \widehat{f}(k). \tag{8} $$

After applying the formula (7), in the Laplace space, the space–time FDE (2) appears in the form

$$ s^\alpha \widetilde{u}(x,s) - s^{\alpha-1} u(x,0) = K \frac{\partial^\beta \widetilde{u}(x,s)}{\partial |x|^\beta} \tag{9} $$