



Efficiency of star-like graphs and the Atlanta subway network



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HIGHLIGHTS

- We provide a general result establishing the global efficiency for all subdivided stars graphs.
- We determine the global efficiency of the Metropolitan Atlanta Rapid Transit Authority (MARTA).
- The MARTA network is about 82% as efficient as a network where stations are connected by a direct line.
- Our conclusion is that subdivided star networks can be efficient models for subways.

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ABSTRACT

The distance $d(i, j)$ between any two vertices i and j in a graph is the number of edges in a shortest path between i and j . If there is no path connecting i and j , then $d(i, j) = \infty$. In 2001, Latora and Marchiori introduced the measure of efficiency between vertices in a graph (Latora and Marchiori, 2001) [1]. The efficiency between two vertices i and j is defined to be $\epsilon_{i,j} = \frac{1}{d(i,j)}$. In this paper, we investigate the efficiency of star-like networks, and show that networks of this type have a high level of efficiency. We apply these ideas to an analysis of the Metropolitan Atlanta Rapid Transit Authority (MARTA) Subway system, and show this network is 82% as efficient as a network where there is a direct line between every pair of stations.

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1. Introduction

The distance $d(i, j)$ between any two vertices i and j in a graph is the number of edges in a shortest path between i and j . If there is no path connecting i and j , then $d(i, j) = \infty$. In 2001, Latora and Marchiori introduced the measure of efficiency between vertices in a graph [1]. The (unweighted) *efficiency* between two vertices i and j is defined to be $\epsilon_{i,j} = \frac{1}{d(i,j)}$ for all $i \neq j$. The *global efficiency* of a graph $E_{glob}(G) = \frac{1}{n(n-1)} \sum_{i \neq j} \epsilon_{i,j}$ which is simply the average of the efficiencies over all pairs of the distinct n vertices.

In 2002, Latora and Marchiori explored the global efficiency of the Boston Subway (MBTA) and found that the MBTA network is 63% as efficient as a network where there is a direct line between any two stations [2]. Motivated by the design of the Metropolitan Atlanta Rapid Transportation Authority (MARTA) Subway network (see Fig. 6), we investigate the global efficiency of subdivided stars. We show that networks of this type have a high level of efficiency. We apply these ideas to an analysis of the MARTA Subway system and show that their network is 82% as efficient as a network where there is a direct line connecting each pair of stations.

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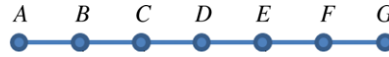
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2. Efficiency

Let P_n denote the path on vertices $v_1, v_2, \dots, v_n$ with edges $v_1v_2, v_2v_3, \dots, v_{n-1}v_n$. The distance $d(v_i, v_j)$ between distinct vertices v_i and v_j is $|i - j|$. Hence the efficiency between v_i and v_j is $\epsilon(v_i, v_j) = \frac{1}{d(v_i, v_j)} = \frac{1}{|i - j|}$.

We present an example using a path with 7 vertices.

Example 1. Let $G = P_7$ with vertices A, B, C, D, E, F and G .



The distances between each pair of vertices are given in the matrix shown below:

$$DM = \begin{array}{c|ccccccc} L(G) & A & B & C & D & E & F & G \\ \hline A & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ B & 1 & 0 & 1 & 2 & 3 & 4 & 5 \\ C & 2 & 1 & 0 & 1 & 2 & 3 & 4 \\ D & 3 & 2 & 1 & 0 & 1 & 2 & 3 \\ E & 4 & 3 & 2 & 1 & 0 & 1 & 2 \\ F & 5 & 4 & 3 & 2 & 1 & 0 & 1 \\ G & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{array}$$

The efficiency matrix is then as follows:

$$EM = \begin{array}{c|ccccccc} E(G) & A & B & C & D & E & F & G \\ \hline A & 0 & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ B & 1 & 0 & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ C & \frac{1}{2} & 1 & 0 & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ D & \frac{1}{3} & \frac{1}{2} & 1 & 0 & 1 & \frac{1}{2} & \frac{1}{3} \\ E & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 & 0 & 1 & \frac{1}{2} \\ F & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 & 0 & 1 \\ G & \frac{1}{6} & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 & 0 \end{array}$$

We note that the matrix is symmetric about the main diagonal. We can also sum the elements in the upper triangle of the matrix: $6(1) + 5(\frac{1}{2}) + 4(\frac{1}{3}) + 3(\frac{1}{4}) + 2(\frac{1}{5}) + 1(\frac{1}{6})$. Finally we divide by the number of non-diagonal elements. Therefore $E_{glob}(P_7) = \frac{1}{7 \cdot 6} \cdot 2 \left(\sum_{i=1}^{7-1} \frac{7-i}{i} \right) = \frac{223}{420}$.

Theorem 2. $E_{glob}(P_n) = \frac{2}{n \cdot (n-1)} \left(\sum_{i=1}^{n-1} \frac{n-i}{i} \right)$.

Proof. Without loss of generality we assume the starting vertex is located to the left of the ending vertex. Note that this will only account for half of the efficiencies. If we want to move i vertices to the right there are only $n - i$ starting vertices. Hence for the efficiency matrix of P_n , there are $n - i$ pairs of vertices whose efficiency is $\frac{1}{i}$. Hence by doubling our efficiencies, we have $E_{glob}(P_n) = \frac{2}{n \cdot (n-1)} \left(\sum_{i=1}^{n-1} \frac{n-i}{i} \right)$. ■

As expected, the global efficiency of a path will vary inversely to the number of vertices. We state this formally in our next theorem.

Theorem 3. $\lim_{n \rightarrow \infty} E_{glob}(P_n) = \lim_{n \rightarrow \infty} \frac{2}{n \cdot (n-1)} \left(\sum_{i=1}^{n-1} \frac{n-i}{i} \right) = 0$.

Proof. For the sake of completeness, we include details. We recall the Mascheroni constant

$$\gamma = \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n \frac{1}{i} - \ln n \right), \quad \text{where } 0 < \gamma < 0.59 \text{ [3].}$$

Rearranging yields $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{i} = \gamma + \lim_{n \rightarrow \infty} \ln n$. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{2}{n \cdot (n-1)} \left(\sum_{i=1}^{n-1} \frac{n-i}{i} \right) &= \lim_{n \rightarrow \infty} \frac{2}{n \cdot (n-1)} \left(\sum_{i=1}^{n-1} \left(\frac{n}{i} - 1 \right) \right) \\ &= \lim_{n \rightarrow \infty} \frac{2}{n \cdot (n-1)} \left(\left(\sum_{i=1}^{n-1} \frac{n}{i} \right) - (n-1) \right). \end{aligned}$$

Since as $n - 1 \rightarrow \infty$, $n \rightarrow \infty$ the upper term on the summation can be changed from $n - 1$ to n .

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