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Relations between allometric scalings and fluctuations in complex systems: The case of Japanese firms

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ABSTRACT

To elucidate allometric scaling in complex systems, we investigated the underlying scaling relationships between typical three-scale indicators for approximately 500,000 Japanese firms; namely, annual sales, number of employees, and number of business partners. First, new scaling relations including the distributions of fluctuations were discovered by systematically analyzing conditional statistics. Second, we introduced simple probabilistic models that reproduce all these scaling relations, and we derived relations between scaling exponents and the magnitude of fluctuations.

1. Introduction

In physiology and anatomy, "allometric scalings" are empirical power laws among percentiles related to size. For example, the brain mass of mammals scales as the corresponding body mass to the power about 0.7 [1]. One of the most famous laws in this field is that between body mass and metabolic rate (i.e., the speed of metabolism), which has the scaling exponent 2/3 [2]. From the viewpoint of statistical physics, this nontrivial scaling relation is explained by the geometric structure of vessel networks and an assumption regarding minimum energy consumption [3].

Recently, allometric scalings have been observed in the real world in various complex systems other than biological systems, and there are many attractive societal applications; for example, economic indices as a function of urban population [4,5] energy consumptions vs. urban population [6], surface area of roads vs. that of cities [7], economic indices vs. national populations [8] or the number of trades vs. the total volumes, etc., for time series of the Spanish stock market [9].

Fluctuations associated with these scalings in complex systems have also been studied. In these studies, the distribution of growth rates is one of the main topics, and the width of growth rates (e.g., the standard deviation or the interquartile distance) vs. system size has been found to follow a power law with a negative exponent [10,11]. In accordance with this scaling, the conditional distributions of growth rates normalized by the widths or the standard deviations conditioned by the system size collapse onto universal curves, which are independent of system size. Such conditional distributions of growth rates have been reported for sales of business firms [10,12,13], national gross domestic products [13,14], total exports, total imports, foreign debts [14], university research activities [15], citations to scientific journals[16], the circulation of magazines and newspapers [17], religious activities [18], bird populations [19] and the metabolic rates of animals [11] etc. This characteristic is also commonly observed between the metabolic rates of animals and business firms [11].

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Here, we focus on the statistical properties of business firms and regard each firm as a typical complex system consisting of various elements such as employees, facilities, and money. Firm activity, in the form of financial reports, is rendered numerically observable. The data within typical financial reports contains many quantities relating to firm size, which we can roughly categorize into three families:

1. Flow variables; such as annual sales, profit, incomes, or tax payments.
2. Stock variables; such as the number of employees, number of branches, or number of factories.
3. Business relations; such as the number of business partners or number of affiliated firms.

Quite interestingly, one body of statistics based on these quantities is generally approximated by a power law distribution that is typically independent of country and observation year; namely, the universal Zipf law for annual sales or profits [20,12,21]. There have been many attempts, typically based on mathematical toy models based on stochastic scale-free dynamics, to clarify why such a power law should hold for a one-body distribution [22].

A few pioneering works exists on allometric scaling of business firms. For example, Fujiwara et al. reported that employee numbers and incomes scale with the corresponding universal conditional distribution for Japanese business firms (up to intermediate size) [23], Watanabe et al. have also analyzed these financial scalings by using the production function [24] and Saito et al. have showed a scaling relationship between numbers of business partners and annual sales [25].

In this study we analyze two- and three-body statistics of typical business variables from the three data categories of annual sales, number of employees, and number of business partners. In particular, we focus on the relation between the scalings among the three quantities and the fluctuations associated with them. By analyzing data from about 500,000 Japanese firms, we find in Section 2 that some pairs of these quantities follow power laws. In addition, we show that the distribution functions for different parameters converge to a unique scaling function through these scaling relations of conditional medians. In the same section, we also find, for three-body relations, scalings of the conditional median of sales and employees as a function of the other two variables. In Section 3, we introduce simple stochastic models that reproduce the all empirical scalings and discuss the relations between these scalings and fluctuations. Finally, we conclude with a discussion in Section 4.

2. Data analysis

The data set was provided by the governmental research institute RIETI (Research Institute of Economy, Trade and Industry) and was based on data collected by Tokyo Shoko Research, Ltd. (TSR) for 2005. It contains approximately one million firms covering practically all active firms in Japan. For each firm, the data set contains various flow variables, stock variables, and a list of business partners categorized into suppliers and customers [26]. From this list, we count the total number of business partners, by superposing all business interactions. We focus on the three basic scale indicators of firms from the three categories: sales s, number of employees l, and number of business partners, which we call the degree k. We neglect those firms for which the three data are not available, thus the number of firms we analyze is 529,291.

2.1. Correlations between two variables

In general, all information regarding three-body statistics for stochastic variables $\{X, Y, Z\}$ is contained in the three-body probability density function (PDF), $P(X, Y, Z)$. To clarify the structure of this function, using the definition of the conditional probability, we decompose it into the three density functions as $P(X, Y, Z) = P(X|Y, Z)P(Y|Z)P(Z)$, where we denote the conditional probability density of Y for given values of Z by $P(Y|Z)$, and where $P(X|Y, Z)$ is the conditional probability density of X for simultaneously given values of Y and Z. We pay attention to the properties of these conditional probability densities. Firstly, we are going to observe the probability densities conditioned by one variable, $P(Y|Z)$, and then the probability densities conditioned by two variables $P(X|Y, Z)$.

We begin by analyzing the two-body relations between the number of employees l and degree k. Fig. 1(a) shows the log–log plot of the number of employees as a function of degree k. We find that all such plots have similar forms for the 5th, 25th, 50th (equivalent to the median), 75th, and 95th percentiles of the number of employees l for a given degree k. In Fig. 1(b), we shift these plots along the vertical axis so that they all lie on the median plot at $k = 100$. All these conditional percentile curves essentially coincide with each other. In particular, for $k \geq 30$, this relation can be described by the following scaling relation:

$$\langle l|k\rangle_q = B_q^{(l|k)} \cdot k^{\gamma_{l|k}} \quad (q = 0.05, 0.25, 0.5, 0.75, 0.95), \tag{1}$$

where $\gamma_{l|k} = 1.0$, $\langle l|k\rangle_q$ is the $100q$ conditional percentile of l given k and $B_q^{(l|k)}$ is a proportional constant for percentile $100q$. The values of $B_q^{(l|k)}$ are estimated to be 0.3 for the 5th percentile, 0.7 for the 25th percentile, 1.6 for the 50th percentile, 4.0 for the 75th percentile and 12 for the 95th percentile. $B_q^{(l|k)}$ can be interpreted as the number of employees per business partner. We find the typical value at the median is 1.6. According to these percentile scaling relations, the PDF of l for a given value of k, $P(l|k)$, is

$$P(l|k) = \frac{1}{f_1(k)} \cdot \Psi_1\left(\frac{l}{f_1(k)}\right), \tag{2}$$