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Quantum Prisoner's Dilemma game on hypergraph networks

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ABSTRACT

We study the possible advantages of adopting quantum strategies in multi-player evolutionary games. We base our study on the three-player Prisoner's Dilemma (PD) game. In order to model the simultaneous interaction between three agents we use hypergraphs and hypergraph networks. In particular, we study two types of networks: a random network and a SF-like network. The obtained results show that in the case of a three-player game on a hypergraph network, quantum strategies are not necessarily stochastically stable strategies. In some cases, the defection strategy can be as good as a quantum one.

1. Introduction

Game theory is a branch of mathematics broadly applied in a great number of fields, from biology to social sciences and economics. A great deal of effort has gone into the study of evolutionary games on graphs, which was initiated by the work of Nowak and May [1]. Since their work was published, a lot of effort was put into studying the problem [2].

Quantum game theory [3] allows the agents to use quantum strategies. The set of quantum strategies is much larger than a classical one; hence it offers the possibility for much more diverse behavior of agents in the network. It has been shown that if only one player is aware of the quantum nature of the system, he/she will never lose in some types of games [4]. Recently, it has been demonstrated that a player can cheat by appending additional qubits to the quantum system [5].

Combining evolutionary games and quantum game theory has resulted in absorbing results [6]. In some cases the quantum strategies can dominate the entire network, infecting it effectively. In our work we like to focus on introducing additional strategies which the agents can use, since in the multi-player case there exists a Pareto-optimal Nash equilibrium for the Prisoner's Dilemma game [7]. Moreover, the PD game is interesting to study, because it was realized experimentally [8].

A hypergraph is a concept that generalizes the concept of a graph by allowing edges to connect more than two nodes at once. This concept can be applied for systems with evolution described by the extended spin-1/2 chain [9]. Based on this, the notion of hypergraphs was proven to be a useful tool for assuring controllability of multipartite quantum systems [10].

This paper is organized as follows. In Section 2 a short review of quantum game theory is given and in Section 3 the types of 3-hypergraph networks used in simulations are described. Section 4 introduces the three-player Prisoner's Dilemma game. In Section 5 the simulation setup is described. Section 6 contains results obtained from computer simulations and their discussion. Finally, in Section 7 the final conclusions are drawn.

2. Quantum game theory

Games that admit the player to use the peculiarities of quantum phenomena are referred to as quantum games [3,11,4]. Of course, they are games in the "classical" sense. Actually, any quantum system which can be manipulated by at least one

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party and where the utility of the moves can be reasonably quantified, may be conceived as a quantum game. To be more specific, a *two-player quantum game* $\Gamma = (\mathcal{H}, \rho, S_A, S_B, P_A, P_B)$ is completely specified by the underlying Hilbert space $\mathcal{H}$ of the physical system, the initial state $\rho \in \mathcal{S}(\mathcal{H})$, where $\mathcal{S}(\mathcal{H})$ is the associated state space, the sets S_A and S_B of permissible quantum operations of the two players, and the payoff (utility) functions P_A and P_B, which specify the payoff for each player. A *quantum strategy* $s_A \in S_A, s_B \in S_B$ is a quantum operation, that is, a completely positive trace-preserving map mapping the state space on itself. The quantum game's definition may also include certain additional rules, such as the order of the implementation of the respective quantum strategies. The generalization to the multi-player case is straightforward. Schematically we have:

$$\rho \mapsto (s_A, s_B) \mapsto \sigma \Rightarrow (P_A, P_B).$$

The following concepts are commonly used in the context of quantum game theory. These definitions are fully analogous to the corresponding definitions in "classical" game theory. A quantum strategy s_A is called the *dominant strategy* of Alice if

$$P_A(s_A, s'_B) \geq P_A(s'_A, s'_B) \tag{1}$$

for all $s'_A \in S_A, s'_B \in S_B$. Analogously we can define a dominant strategy for Bob. A pair (s_A, s_B) is said to be an *equilibrium in dominant strategies* if s_A and s_B are the players' respective dominant strategies. A combination of strategies (s_A, s_B) is called a *Nash equilibrium* if

$$P_A(s_A, s_B) \geq P_A(s'_A, s_B), \tag{2}$$

$$P_B(s_A, s_B) \geq P_B(s_A, s'_B). \tag{3}$$

A pair of strategies (s_A, s_B) is called *Pareto-optimal*, if it is not possible to increase one player's payoff without lessening the payoff of the other player. A solution in dominant strategies is the strongest solution concept for a non-zero sum game. In the Prisoner's Dilemma [4,3]:

	Bob : C	Bob : D
Alice : C	(3, 3)	(0, 5)
Alice : D	(5, 0)	(1, 1)

(the numbers in parentheses represent the row (Alice) and column (Bob) player's payoffs, respectively). Defection is the dominant strategy, as it is favorable regardless of what strategy the other party chooses. In a Nash equilibrium neither player has a motivation to unilaterally alter his/her strategy, as this action will not increase his/her payoff. Given that the other player will stick to the strategy corresponding to the equilibrium, the best result is achieved by also playing the equilibrium solution. The concept of Nash equilibrium is therefore of paramount importance. However, it is only an acceptable solution concept if the Nash equilibrium is not unique. For games with multiple equilibria we have to find a way to eliminate all but one of the Nash equilibria. A Nash equilibrium is not necessarily efficient. We say that an equilibrium is Pareto-optimal if there is no other outcome which would make all players better off. Up to now a lot of papers on quantum games have been published and some applications outside the field of physics have also been discussed [12–18].

3. Hypergraphs and hypergraph networks

We assume a hypergraph [19] network $H(X, E)$ where X is a set of nodes and E is a set of non-empty subsets of X, $E \subseteq 2^X$. Elements of E are the hyperedges of H. We keep within the boundaries of the case when every subset of X, $A \in E$ satisfies $|A| = 3$, i.e. every edge of the hypergraph connects three nodes exactly. Hereafter we will refer to this structure as a 3-hypergraph. We set $N = |X|$—the total number of agents.

We construct two types of networks: a random network, in which all hyperedges connect random nodes and a SF-like [20] network. We set the number of hyperedges in the random case to $|E| = 10{,}000$. The SF-like network is constructed in the following way: First, a network of $m_0 \ll N$ all connected nodes is created. Then a new node with $m < m_0$ links is added to the network. For each of the m links, a pair of unique nodes is chosen from the existing network and a new hyperedge is added. The probability of a node i being chosen is given by:

$$p_{sf}(i) = \frac{k_i}{\sum_{j \in X} k_j}, \tag{4}$$

where k is the degree of a node. This procedure is repeated until the number of nodes of the network reaches N.

4. Three-player PD game

The classical Prisoner's Dilemma game is as follows: two players can either cooperate (C) or defect (D). When they both cooperate, each receives a payoff of 3. On the other hand, when they both defect, each receives a payoff of 1. When one defects, he/she receives a payoff of 5, while the other gets 0.

This approach can be extended to a greater number of players. In the three-player case, the payoff matrix is shown in Table 1. We can see that every player is better off defecting than cooperating no matter what the other players do. In terms