Physica A 391 (2012) 456–463

Contents lists available at SciVerse ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

A semi-discrete model and its approach to a solution for a wide moving jam in traffic flow

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ARTICLE INFO

Article history:
Received 1 March 2011
Received in revised form 7 August 2011
Available online 24 August 2011

Keywords:
Lagrange coordinates
Semi-discrete model
Car-following model
Wide moving jam

ABSTRACT

This paper investigates the analytical and numerical solutions to wide moving jams in traffic flow. Under the framework of the Lagrange coordinates, a semi-discrete model and a continuum model correlate with each other, in which the former model approaches the latter as the increment ΔM in the former model vanishes. This implies that the solution to a wide moving jam in the latter model, which can be analytically derived using the known theory, can be conceivably taken as an approximation to that of the former model. These results were verified through numerical simulations. Because a detailed understanding of the traffic phase "wide moving jam" is very important for the further development of Kerner's three-phase traffic theory, this study helps to explain the empirical features of traffic breakdown and resulting congested traffic patterns that are observed in real traffic.

1. Introduction

Kerner [1,2] introduced a "three-phase traffic theory" in which there are two phases in congested traffic: 1. Synchronized flow and 2. Wide moving jam. This theory explains the traffic breakdown and other empirical features of traffic flow that were observed on different highways in various countries. In this three-phase theory, the traffic breakdown is described as a first-order phase transition from free flow to synchronized flow ($F \to S$ transition), whereas wide moving jams are formed through a sequence of two phase transitions called $F \to S \to J$ transitions: Firstly a $F \to S$ transition occurs and later and usually at another road location a $S \to J$ transition is realized.

A wide moving jam with $F \to J$ transition was first discovered by Kerner and Konhauser [3] based on a numerical and analytical study of a version of Payne's model [4]. Later, the results of [3] about wide moving jams have been incorporated and further developed in many traffic flow models, e.g., the characteristic features of wide moving jam phase in cellular automaton models [5,6] and higher-order models [7–13]. Although most of these studies cannot show synchronized flow, thus being unable to predict $F \to S \to J$ transitions, they are important for the further development of traffic flow theory, because the characteristic features of wide moving jams play a very important role in Kerner's three-phase traffic theory. On the other hand, most semi-discrete (car-following) models (e.g., in Refs. [14–16]) failed to analytically show the characteristic features of a wide moving jam even through they are deterministic. This could be probably due to the unavailability of analytical solutions to most non-linear ordinary differential systems.

This paper studies the characteristic features of wide moving jams in a semi-discrete model. Using the concept of the Lagrange coordinates, the formulated semi-discrete model with an increment ΔM could converge to a continuum higher-order model for $\Delta M \to 0$, where ΔM is the mass between two adjacent particles. This implies consistency between the

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doi:10.1016/j.physa.2011.08.035

semi-discrete and continuum models. Therefore, with the error of $O(\Delta M)$ the characteristic parameters of a wide moving jam in the former model can be well approximated by those in the latter model, which can be similarly derived through application of the weak solution theory of hyperbolic conservation laws, as was shown in Zhang et al. [11–13]. To verify the convergence, we numerically demonstrate that the semi-discrete model is able to reproduce a regular wide moving jam, and that for the refinement of ΔM its characteristic parameters do approach those that are analytically derived through the continuum model. Even with a large value of $\Delta M = 1$, in which the semi-discrete model reduces to a car-following model, the approximation is also very good.

Here, we mention a similar establishment between the continuum and the car following models in Aw et al. [17] and Greenberg [8,18], in which the convergence of the latter solution to the former solution was generally and indirectly shown by "shrinking" the time and space coordinates such that the length of a car approaches zero [17]. More relevantly, Greenberg's analytical study of traveling waves that was based on different assumptions should be suited more generally for a narrow and wide moving jam [8]. However, he did not show convergence and a comparison between analytical and numerical solutions. In this regard, the present paper also serves as a supplement to the aforementioned studies. We should also note that the formulation under the Lagrange coordinate system mostly results in "anisotropic" models, which are related to those discussed under the Euler coordinate system, e.g., in Refs. [9,10,19–22].

The remainder of the paper is organized as follows. In Section 2, we discuss the basic concepts of one-dimensional fluid in relation to the Lagrange coordinates, based on which we formulate an acceleration equation for traffic flow. In Section 3, the linear stability condition of the equilibrium solution for the resulting continuum model is derived and an analytical wide moving jam solution is constructed. In Section 4, the semi-discrete model is naturally formulated based on the discussion in Section 2, and the aforementioned convergence of its solution for a wide moving jam to that of the continuum model is demonstrated through numerical simulations. We conclude the paper in Section 5.

2. General discussion of model equations

In most studies, the initial position of a particle is more generally taken as the Lagrange coordinates of the particle. However, for a one-dimensional continuum like traffic flow, the total mass upstream of a particle also remains unchangeable and thus is more conveniently used as the Lagrange coordinate to identify the particle. In this case, equations including the mass conservation and acceleration are easily established, and it is straightforward to derive the motion equations of particles by direct discretization of these equations.

2.1. Lagrange coordinates and mass conservation

Let $M(x, t)$ denote the total mass not passing through position x at time t in a one-dimensional continuum. Then, the density of the fluid is defined as

$$\rho(x, t) = \lim_{\Delta x \to 0} \frac{M(x + \Delta x, t) - M(x, t)}{\Delta x} = M_x(x, t), \tag{1}$$

and the flow is defined as

$$q(x, t) = -\lim_{\Delta t \to 0} \frac{M(x, t + \Delta t) - M(x, t)}{\Delta t} = -M_t(x, t). \tag{2}$$

Here, the mass M measures the quantity of substance in the continuum, which is denoted as the number of cars in traffic flow. Application of the identity $M_{xt} = M_{tx}$ to Eqs. (1) and (2) in a smooth solution region gives rise to the mass conservation:

$$\rho_t + q_x = 0.$$

The formulation becomes the same as that in Refs. [23,24] by replacing $M(x, t)$ with $-N(x, t)$, where $N(x, t)$ is the total mass passing through location x at time t.

Under the continuum hypothesis, the total mass not passing through a specific particle remains a constant in the motion or is independent of x and t because no overtaking is allowed. Moreover, for fixed t, Eq. (1) implies that M is strictly increasing of x as the vacuum is not considered or would be specially treated. Therefore, it is convenient to identify a specific particle in the motion by the index M (instead of x) and time t, which constitutes the Lagrange coordinate system under Eqs. (1) and (2). Accordingly, we rewrite Eq. (1) as

$$s(M, t) = x_M(M, t), \tag{3}$$

where $s(M, t) = (\rho(x, t))^{-1}$ is called the specific volume at (x, t) or (M, t), and $x(M, t)$ is the position of particle M at time t. The motion speed of particle M at time t is defined as

$$u(M, t) = \lim_{\Delta t \to 0} \frac{x(M, t + \Delta t) - x(M, t)}{\Delta t} = x_t(M, t). \tag{4}$$