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Generalized Langevin equation driven by Lévy processes: A probabilistic, numerical and time series based approach

Ary V. Medino [a,*], Sílvia R.C. Lopes [b], Rafael Morgado [c], Chang C.Y. Dorea [a]

[a] Department of Mathematics, University of Brasília, 70910-900, Brasília-DF, Brazil
[b] Mathematics Institute, Federal University of Rio Grande do Sul, 91509-900, Porto Alegre - RS, Brazil
[c] University of Brasília, 70910-900, Brasília-DF, Brazil

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A B S T R A C T

Lévy processes have been widely used to model a large variety of stochastic processes under anomalous diffusion. In this note we show that Lévy processes play an important role in the study of the Generalized Langevin Equation (GLE). The solution to the GLE is proposed using stochastic integration in the sense of convergence in probability. Properties of the solution processes are obtained and numerical methods for stochastic integration are developed and applied to examples. Time series methods are applied to obtain estimation formulas for parameters related to the solution process. A Monte Carlo simulation study shows the estimation of the memory function parameter. We also estimate the stability index parameter when the noise is a Lévy process.

1. Introduction

The use of Lévy processes and stable distributions to model complex systems constitutes a rich area of research. In recent decades, the interest in nonnormal probability models has grown considerably in several fields of science. Applications can be found in laser cooling [1], turbulent flow [2], channeling in crystals [3], evolution of stock prices [4–6], protein diffusion structures [7], optimization and search problems [8], human travel [9], etc. Frequently in these models, solutions to stochastic differential equations are searched and these solutions can be cast as stochastic integrals. In this context, the GLE (Generalized Langevin Equation) driven by Lévy processes may arise as a model to such phenomena. The main difficulty relies on the fact that Itô's formula cannot be used, since the stochastic process does not have finite second moment. To overcome this problem, we propose a solution to the GLE using stochastic integration in the sense of convergence in probability.

In Statistical Mechanics, the GLE

$$V'(t) = -\int_0^t \Gamma(t-s)V(s)\mathrm{d}s + \eta(t), \quad V(0) = V_0 \tag{1.1}$$

is used to study the movement of a Brownian particle immersed in a fluid and subject to random collisions with molecules of the fluid [10,11]. In this equation, $V = (V(t), t \geq 0)$ and $\eta = (\eta(t), t \geq 0)$ are stochastic processes which represent respectively the velocity of the particle and the random force acting on it caused by the collisions. The process η is called *fluctuation* or *noise* and Γ, called *memory function*, is a deterministic function.

* Corresponding author. Tel.: +55 61 31076435.
E-mail addresses: A.V.Medino@mat.unb.br, arymedino@unb.br (A.V. Medino).

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When the memory function is a δ Dirac function, i.e. $\Gamma(t) = \gamma\delta(t)$ where γ is a constant and η is the white noise, the GLE becomes the classical Langevin equation

$$V'(t) = -\gamma V(t) + \eta(t), \quad V(0) = V_0. \tag{1.2}$$

A formal way to study Eq. (1.2) makes use of Laplace transforms, as is done in Refs. [10] or [11]. Under this approach, the solution is given by the process

$$V(t) = V_0 e^{-\gamma t} + \int_0^t e^{-\gamma(t-s)}\eta(s)ds. \tag{1.3}$$

Alternatively, a rigorous mathematical study of the classical Langevin equation requires the use of Itô's stochastic calculus [12]. In this case, V satisfies the stochastic differential equation

$$dV(t) = -\gamma V(t)dt + \beta dW(t), \quad V(0) = V_0, \tag{1.4}$$

where $W = (W(t), t \geq 0)$ is the Wiener process, also called Standard Brownian motion. Eq. (1.4) is just a differential representation of the integral equation

$$V(t) = V_0 + \int_0^t -\gamma V(s)ds + \int_0^t \beta dW(s), \tag{1.5}$$

where in the second integral we have Itô's integral of the white noise, $\eta(t)dt = \beta dW(t)$. Eq. (1.4), or equivalently (1.5), can be solved applying Itô's formula and the solution process, called the Ornstein–Uhlenbeck process, is given by

$$V(t) = V_0 e^{-\gamma t} + \int_0^t e^{-\gamma(t-s)}dW(s). \tag{1.6}$$

Assuming that all the processes are of second order, that is, they have finite quadratic mean, Kannan in Ref. [13] studied a subclass of GLE. Solutions were obtained using the Bochner integral [14] and the notion of derivative in Hilbert spaces. It was shown that any *mean square solution*, $V = (V(t), t \geq 0)$, of the GLE has the form

$$V(t) = V_0\rho(t) + \int_0^t \rho(t-s)\eta(s)ds, \tag{1.7}$$

where ρ is a deterministic function satisfying the Volterra integro-differential equation

$$\rho'(t) = -\int_0^t \Gamma(t-s)\rho(s)ds, \quad \rho(0) = 1. \tag{1.8}$$

Note that when $\Gamma(t-s) = -\gamma\delta(t-s)$, the integro-differential equation (1.8) leads to $\rho(t) = e^{-\gamma t}$ and Kannan's representation (1.7) corresponds to Eq. (1.3), equivalently, to the Ornstein–Uhlenbeck process (1.6), with $\eta(s)ds = \beta dW(s)$.

Dropping the hypothesis of finite quadratic mean, neither Itô's stochastic calculus nor Kannan's approach can be applied. Nevertheless, (1.3) and (1.6)–(1.8) suggest that solutions weaker than *mean square solution* can be derived. The main goal of this work is to handle this task by using a weaker concept of stochastic integration, called stochastic integration in the sense of convergence in probability. Our approach is presented in Section 2 and more theoretical details are presented in the Appendix.

Potential applications of our approach can be found in the GLE based modeling of anomalous diffusions, a sort of phenomenon observed in some physical systems. That is, if

$$X(t) = \int_0^t V(s)ds \tag{1.9}$$

is the position of a particle with velocity $V = (V(t), t \geq 0)$, then the quadratic mean displacement $\mathbb{E}[X^2(t)]$ does not grow linearly as time $t \to \infty$. See, for instance, [15–18], where the GLE is used to study anomalous diffusions. In this case, to handle processes with diverging moments, we introduced in Ref. [16] a generalization of the *anomalous diffusion index* that is suitable to use in cases of Lévy motions. For instance, it is possible to show, under mild conditions, that the moment $m(t) = \mathbb{E}[V(t)]$ is given by the convolution equation

$$\mu\rho(t) = m'(t) + m\int_0^t \Gamma(t-s)\rho(s)ds, \tag{1.10}$$

where $\mu = \mathbb{E}[L(1)]$ and $m = \mathbb{E}[V_0]$, see [19] for details. We just briefly mention that up to now we have not been able to overcome some mathematical difficulties trying to perform derivations to obtain expressions for the moments of fractional order. Extensions of this work related to this point and also to Lévy flights subject to external force fields are expected. We refer the reader to the paper [20] for insights.