Physica A 391 (2012) 767–778

Contents lists available at SciVerse ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

The rank-size scaling law and entropy-maximizing principle

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ARTICLE INFO

Article history:
Received 29 April 2011
Received in revised form 2 July 2011
Available online 28 July 2011

Keywords:
Urban system
Rank-size rule
Zipf's law
Fractals
Fractal dimension
Scaling range
Hierarchy
Entropy-maximizing method

ABSTRACT

The rank-size regularity known as Zipf's law is one of the scaling laws and is frequently observed in the natural living world and social institutions. Many scientists have tried to derive the rank-size scaling relation through entropy-maximizing methods, but they have not been entirely successful. By introducing a pivotal constraint condition, I present here a set of new derivations based on the self-similar hierarchy of cities. First, I derive a pair of exponent laws by postulating local entropy maximizing. From the two exponential laws follows a general hierarchical scaling law, which implies the general form of Zipf's law. Second, I derive a special hierarchical scaling law with the exponent equal to 1 by postulating global entropy maximizing, and this implies the pure form of Zipf's law. The rank-size scaling law has proven to be one of the special cases of the hierarchical scaling law, and the derivation suggests a certain scaling range with the first or the last data point as an outlier. The entropy maximization of social systems differs from the notion of entropy increase in thermodynamics. For urban systems, entropy maximizing suggests the greatest equilibrium between equity for parts/individuals and efficiency of the whole.

1. Introduction

If the number of cities in a region is large enough, the city-size distribution usually follows the well-known Zipf's law [1]. Zipf's law is associated with the rank-size rule, and the former does not differ from the latter in practice. Formally, the general form of the rank-size scaling law can be expressed as

$$P_k = P_1 k^{-q}, \tag{1}$$

where k denotes the rank by the size of the cities in the region ($k = 1, 2, 3, \ldots$), P_k refers to the population size of the kth city, P_1 to the population of the largest city, and q, the scaling exponent of the rank-size distribution. Empirically, q values always approach 1 [2–10,1]. If $q = 1$, then Eq. (1) becomes the pure form of Zipf's law [11]. The rank-size regularity is associated with fractals, and the power exponent indicates the fractal dimension of city-size distributions [12–19]. In fact, the Zipf ranking approach can be applied to a wide range of systems, particularly economic systems [20–26,1]. The city-size distribution is only one of the typical phenomena following the rank-size scaling law.

In theory, if we define a self-similar hierarchy of M levels, with $f_1 = 1$ city at the first level, $f_2 = r_f$ cities at the second level, $f_3 = r_f^2$ cities at the third level, and so on, the rank-size law, Eq. (1), can be expressed as a pair of exponential laws in the following forms:

$$f_m = f_1 r_f^{m-1}, \tag{2}$$

$$P_m = P_1 r_p^{1-m}, \tag{3}$$

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doi:10.1016/j.physa.2011.07.010

where m denotes the ordinal number of city levels in the hierarchy ($m = 1, 2, 3, \ldots, M$), f_m refers to the number of cities in the mth level (for short, *city number*), P_m to the average size of the f_m cities, P_1 to the average population size of the cities in the top level ($f_1 = 1$), r_f to the city *number ratio* ($r_f = f_{m+1}/f_m$), and r_p, the city *size ratio* ($r_p = P_m/P_{m+1}$). Eq. (2) represents the *number law*, and Eq. (3), the *size law* of cities. If $r_f = r_p = 2$, Eqs. (2) and (3) are equivalent to the 2^n rule of Davis [27]. Therefore, the pair of exponential functions represents the generalized 2^n rule of cities [13].

As stated above, the self-similar hierarchy is based on a top-down order, i.e., from the largest to the smallest. Due to the mirror symmetry of exponential models, we can describe the hierarchy in the inverse order equivalently, that is, from the smallest to the largest [28]. Based on the bottom-up order, the number law Eq. (2) can be re-expressed in an equivalent form such as

$$f_m = f_M r_f^{1-m}, \tag{4}$$

where m denotes the order from the smallest to the largest cities, $f_M = f_1'$ is the city number of the bottom level in the top-down order hierarchy, and f_1', the city number in the first level of hierarchy from bottom to top. Correspondingly, the size law can be rewritten as $P_m = P_M r_p^{m-1}$, in which P_M indicates the average size of the smallest cities seen in the first type of hierarchy. By the generalized 2^n principle, the rank-size scaling law can be reconstructed as a hierarchical scaling law [29]. Obviously, from Eqs. (2) and (3) follows a scaling relation in the form

$$f_m = \eta P_m^{-D}, \tag{5}$$

where $\eta = f_1 P_1^D$ denotes a proportionality coefficient and $D = 1/q$ proves to be a fractal dimension of city rank-size distribution [30].

Zipf's law is in fact one of the scaling laws found both in nature and society [31]. However, for a long time, the rank-size rule was not derivable from the general principle so that no convincing physical and economic explanation could be provided for the existence of the scaling relation and exponent value [32–34]. In order to bring to light the underlying principle of the rank-size scaling of cities, scientists have tried to derive Zipf's law through approaches such as the entropy-maximizing method. It has been shown that the rank-size law is related to the maximum entropy models [35]. Curry [36] once made a derivation of the rank-size rule by means of entropy maximization. However, his demonstration has three drawbacks. First, he actually derived an exponential model associated with Eq. (4) rather than a power law correlated with Eq. (1). Second, he let $f_m = m$, which meant that the, city number in each class is confused with the ordinal number of the class. Third, one of the constraint conditions was set as $f_m P_m = const$, which does not accord with the reality. Anastassiadis [37] attempted to derive Eq. (1) directly, but so many assumptions were made that the mathematical process became too complicated to comprehend. Chen and Liu[38] made another derivation with the entropy-maximizing principle based on Curry's work and the self-similar hierarchy, and they deduced Eqs. (3) and (4), which imply Eqs. (2) and (5). However, they were unable to avoid the second and third weaknesses of Curry's work, which took credence away from their other derivations and explanations.

In fact, urban evolution falls into two major, sometimes contradictory, processes: city number increase in a system of cities and population size growth of each city in the system [39,34]. The former indicates what is called *external complexity* associated with frequency distribution scaling, and the latter, *internal complexity* associated with size distribution scaling. The concepts of external and internal complexity came from biology [40]. In this paper, I will present a new derivation of the rank-size scaling law of cities by employing the entropy-maximizing method, and then propose a new explanation for Zipf's law. First, assuming entropy-maximizing of city frequency distribution, I derived Eq. (4), which is equivalent to Eq. (2). Second, assuming entropy-maximizing of city size distribution, I derived Eq. (3), and from Eqs. (2) and (3) follows Eq. (5), which suggests Eq. (1). Third, assuming entropy-maximizing of both city frequency and population size distributions, I derived the scaling exponent $q = 1$. Two empirical analyses will be made to lend support to the theoretical results. The pivotal work of this paper is to construct a constraint equation and prove it mathematically.

2. Models and derivations

2.1. Derivation of the number law

Suppose there is a region **R** with n number of cities and a total urban population of N within this region. A basic assumption is made as follows: the probability of an urban resident living in different cities is equal [36]. By the average population sizes in different groups, we can classify the cities into M levels in a bottom-up order and form a hierarchy (Fig. 1). If the city number at the mth level is f_m and the mean size of the f_m cities is P_m, the *state number* of the n city distribution in different classes, W, can be expressed as a problem of ordered partition of the city set (i.e., system of cities). In fact, an ordered partition of "type f_1 + type f_2 + $\cdots$ + type f_M" is one in which the mth part has f_m members, for $m = 1, 2, \ldots, M$. The state number of such partitions is given by the following multinomial coefficient:

$$W(f) = \binom{n}{f_1, f_2, \ldots, f_m, \ldots, f_M} = n! \Big/ \prod_{m=1}^{M} f_m!, \tag{6}$$