Physica A 392 (2013) 4036–4047

Contents lists available at SciVerse ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Spectral analysis and generation of certain highly oscillatory curves related to chaos, part 2: Calculation aspects

Goong Chen [a,b], Tingwen Huang [b], Maria Christina Mariani [c,*], Roza Espandeary [b,d], Rym Kanes [b,d], Mohammed Sheriff [b], Srikanth Srinivasan [b,d]

[a] Department of Mathematics and Institute for Quantum Science and Engineering, Texas A&M University, College Station, TX 77843, USA
[b] Science Program, Texas A&M University - Qatar, Education City, Doha, Qatar
[c] Department of Mathematical Sciences, University of Texas at El Paso, El Paso, TX 79968, USA
[d] UREP (Undergraduate Research Experience Program) at Texas A&M University, Qatar

HIGHLIGHTS

- Improves the analysis and understanding of chaos.
- Uses concrete examples to illustrate growth of total variations of iterates of chaotic maps.
- Derives useful formulas for the evaluation of Fourier spectra of piecewise linear maps.
- Shows examples of evaluation of Fourier coefficients of highly oscillatory phenomena.

ARTICLE INFO

Article history:
Received 7 December 2012
Received in revised form 2 April 2013
Available online 3 May 2013

Keywords:
Spectral analysis
Highly oscillatory curves
Chaos

ABSTRACT

The Fourier spectrum of a curve on a given interval, as represented by the Fourier series, is very reflective of the oscillatory behavior of the curve as the "weights" of the Fourier coefficients can indicate the individual prominence of frequencies of oscillations. But, unfortunately, the calculations of the Fourier coefficients can often get quite cumbersome.

In this paper, we first review the oscillatory behavior of chaotic maps as time series, and discuss certain properties between the exponential growth of total variations and the Fourier coefficients. Then we will discuss some simple techniques that can simplify the calculations of Fourier coefficients. In the process, one can also obtain valuable information about the asymptotic properties of the Fourier coefficients.

Published by Elsevier B.V.

1. Introduction

This paper is a continuation [1]. Here, the main emphasis is to show more, concrete computations of the Fourier spectra of highly oscillatory curves.

The daily movement of stocks, commodities, economic data, etc., is often modeled as a discrete nonlinear dynamic process

$$\begin{cases} \boldsymbol{x}_{k+1} = \boldsymbol{f}(\boldsymbol{x}_k; \boldsymbol{\mu}_k), & k = 0, 1, 2, \ldots; \boldsymbol{x}_k \in \mathbb{R}^N, \boldsymbol{\mu}_k \in \mathbb{R}^M, \\ \boldsymbol{x}_0 \in \mathbb{R}^N & \text{is given,} \end{cases} \tag{1}$$

where $\boldsymbol{f}\colon \mathbb{R}^N \to \mathbb{R}^N$, and $\boldsymbol{\mu}_k$ represents the parameter vector which may also change at every step k.

* Corresponding author. Tel.: +1 575 642 5561.
E-mail address: mcmariani@utep.edu (M.C. Mariani).

0378-4371/$ – see front matter. Published by Elsevier B.V.
http://dx.doi.org/10.1016/j.physa.2013.04.041

System (1) is often called a (multidimensional) *finite difference equation*. Mathematical analysis for (1) is extremely challenging, especially in the multidimensional setting. Many such systems exhibit chaotic behavior. The *Smale Horseshoe* technique remains the key method for studying chaos in such systems. But difficulties in the analysis of chaos make the field wide open.

Instead of studying Eq. (1) for *individual* trajectories, we may look at the map $\boldsymbol{f}$ itself. Here, we will only examine the one-dimensional case:

$$\begin{aligned} &x_{k+1} = f(x_k), \quad k = 0, 1, 2, \ldots; x_0 \in \mathbb{R} \text{ is given;} \\ &f\colon \mathbb{R} \to \mathbb{R} \text{ is assumed to be sufficiently smooth,} \end{aligned} \tag{2}$$

in the hope that it can guide us into further study of higher dimensional problems. The iteration in (2) gives

$$x_k = f^k(x_0), \quad k = 1, 2, \ldots,$$

where

$$f^k \overset{\text{def.}}{=} \underbrace{f \circ f \circ \cdots \circ f}_{\substack{k\text{-times}\\ \text{composition}}}, \text{ called the } k\text{-th iterate of } f, \quad k = 1, 2, \ldots.$$

Sometimes, to avoid possible confusion with powers or superscripts, we also write the above as $f^{\textcircled{k}}$. Periodic, nonperiodic, asymptotic, chaotic and other oscillatory behaviors of the trajectory of (2) can be learned from the iterated maps f^k themselves. We recall several theorems; see G. Chen and Y. Huang [2], e.g. In the following, $V_I(f)$ denotes the *total variation* of the map f on the interval I, which is a good index for *quantifying the amount of oscillations* of f on I [2,1,3,4].

For the sake of sufficient self-containedness, we first recall some theorems from [2,4].

Theorem 1 ([5, Main Theorem 8, p. 2180])**.** *Let I be a closed interval and $f\colon I \to I$ be continuous. Assume that f has two fixed points on I and a pair of period-2 points on I. Then*

$$\lim_{n\to\infty} V_I(f^n) = \infty. \quad \square$$

Theorem 2 ([2, Theorem 2.5, p. 24] and [5, Main Theorem 5, p. 2179])**.** *Let I be a bounded interval and let $f\colon I \to I$ be continuous such that f has a period-3 orbit $\{p_1, p_2, p_3\}$ satisfying $f(p_1) = p_2, f(p_2) = p_3$ and $f(p_3) = p_1$. Then*

$$\lim_{n\to\infty} V_I(f^n) \geq K e^{\alpha n} \quad \text{for some } K, \alpha > 0, \tag{3}$$

i.e., the total variation of f^n grows exponentially with n.

Theorem 3 ([2, Theorem 2.6, p. 27])**.** *Let I be a bounded closed interval and $f\colon I \to I$ be continuous. Assume that $I_1, I_2, \ldots, I_n$ are closed subintervals of I which overlap at most at endpoints, and the covering relation*

$$I_1 \longrightarrow I_2 \longrightarrow I_3 \longrightarrow \cdots \longrightarrow I_n \longrightarrow I_1 \cup I_j, \quad \text{for some } j \neq 1, \tag{4}$$

holds. Then for some $K > 0$ and $\alpha > 0$,

$$V_I(f^n) \geq K e^{\alpha n} \longrightarrow \infty, \quad \text{as } n \to \infty. \quad \square \tag{5}$$

Theorem 4 ([6], [3, Theorem 5, p. 1442] and [1, Theorem 2.2, p. 1455])**.** *Let $f \in L^1(I)$ where $I = [0, 1]$. Denote*

$$c_\omega = \int_0^1 e^{-2\pi i\omega x} f(x)dx, \quad \text{for } \omega \in \mathbb{R},$$

and let $S(f)(x)$ denote its Fourier series given by

$$S(f)(x) = \sum_{k=-\infty}^{\infty} c_k e^{i2\pi kx}.$$

Then we have

(1) *The Riemann–Lebesgue Lemma:*

$$\lim_{|\omega|\to\infty} c_\omega = 0;$$

(2) *Dirichlet's Theorem: If f is differentiable at $\xi_0 \in \overset{\circ}{I}$ in the classical sense, then $S(f)(\xi_0) = f(\xi_0)$ in the sense that*

$$\lim_{M,N\to\infty} \sum_{k=-M}^{N} c_k e^{2\pi ik\xi_0} = f(\xi_0); \tag{6}$$