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Noise-intensity fluctuation in Langevin model and its higher-order Fokker–Planck equation

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ABSTRACT

In this paper, we investigate a Langevin model subjected to *stochastic intensity noise* (SIN), which incorporates temporal fluctuations in noise-intensity. We derive a higher-order Fokker–Planck equation (HFPE) of the system, taking into account the effect of SIN by the adiabatic elimination technique. Stationary distributions of the HFPE are calculated by using the perturbation expansion. We investigate the effect of SIN in three cases: (a) parabolic and quartic bistable potentials with additive noise, (b) a quartic potential with multiplicative noise, and (c) a stochastic gene expression model. We find that the existence of noise-intensity fluctuations induces an intriguing phenomenon of a bimodal-to-trimodal transition in probability distributions. These results are validated with Monte Carlo simulations.

1. Introduction

Many real-world systems are inhomogeneous and fluctuant. Stochastic processes are often used for modeling such fluctuant systems in many fields, including physics, biology, and chemistry. The dynamics in these systems can be described by a Langevin equation given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) + g(x)\xi_x(t), \tag{1}$$

where $f(x) = -\partial_x U(x)$, $U(x)$ denotes a potential, $g(x)$ is an arbitrary function of x, and $\xi_x(t)$ is the white Gaussian noise with correlation $\langle \xi_x(t)\xi_x(t')\rangle = 2D_x\delta(t-t')$. Although white noise can reflect microscale properties of fluctuations, it is uniform when seen from mesoscopic or macroscopic time scales (Fig. 1(a)).

One widely used approach for incorporating mesoscopic or macroscopic inhomogeneity in noise is colored noise, where the white noise $\xi_x(t)$ in Eq. (1) is replaced by $z(t)$ with the Ornstein–Uhlenbeck process:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -\frac{z}{\tau} + \frac{\xi_z(t)}{\tau}, \tag{2}$$

where $\xi_z(t)$ expresses the white Gaussian noise [$\langle \xi_z(t)\xi_z(t')\rangle = 2D_z\delta(t-t')$]. Eq. (2) yields colored noise with the finite correlation time τ, that is, $\langle z(t)z(t')\rangle = (D_z/\tau)\exp\left\{-|t-t'|/\tau\right\}$. The existence of correlation time in the noise sources can induce many phenomena, such as resonant activation [1] and noise-enhanced stability [2–6].

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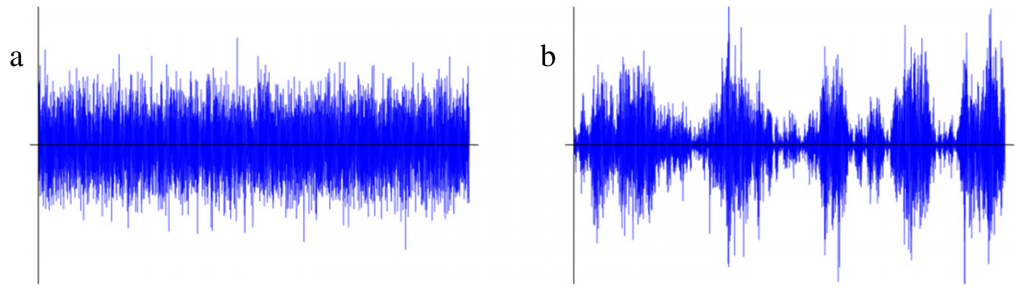

Fig. 1. (Color online) Intuitive paths of (a) white noise and (b) SIN whose intensity is governed by the Ornstein–Uhlenbeck process (Eq. (4)).

In the present paper, we deal with mesoscopic time-scale inhomogeneity in a way other than with colored noise; here, we consider temporal noise-intensity fluctuations. We assume that the noise intensity in Eq. (1) is not constant but modulated by the Ornstein–Uhlenbeck process. Our model is described by the following coupled Langevin equations instead of Eq. (1):

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) + g(x)s\xi_x(t), \tag{3}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = -\gamma(s-\alpha) + \sqrt{\gamma}\xi_s(t). \tag{4}$$

Here, $\xi_s(t)$ is the white Gaussian noise $[\langle\xi_s(t)\xi_s(t')\rangle = 2D_s\delta(t-t')]$, and γ and α are the relaxation rate and the mean of the Ornstein–Uhlenbeck process, respectively. The intensity-modulated noise term $s\xi_x(t)$ in Eq. (3) is herein called *stochastic intensity noise* (SIN) (Fig. 1(b)). This point of view was originally introduced in the Heston stochastic volatility models in financial engineering [7] and has since been analyzed in econophysics [8]. With $f(x) \propto x$ and $g(x) \propto x$, Eqs. (3) and (4) are similar to those in the Heston model, where the variance of noise is governed by the Feller process (also referred to as the square-root process or the Cox–Ingersoll–Ross process). Escape events in the Heston model were studied in Ref. [9] using a cubic potential. Note that the variable s in Eqs. (3) and (4) takes negative as well as positive values since our model can be considered as white noise with multiplicative term s, which is in contrast with the positive variance in the Heston model [7]. In physics, superstatistics includes the concept of temporal and/or spatial environmental fluctuations [10–14]. Superstatistics was originally introduced, under specific conditions, to account for asymptotic power-law distributions (e.g., q-exponential distributions and q-Gaussian distributions) that emerge as maximizers of non-additive (Tsallis) entropy [15,16]. Superstatistics has since been applied to the interpretation of quasi-equilibrium thermodynamics, and concepts of superstatistics have also been applied to stochastic processes [11,13,17–20]. In particular, Ref. [17] extended superstatistics to the path-integral representation and showed that some stochastic models can be covered by this representation. A direct connection between Tsallis statistics and financial stochastic processes was indicated in Ref. [21].

In many biological and chemical processes, the relaxation time of x may be larger than that of environmental fluctuations (noise-intensity processes). This is the case in stochastic gene expression models in which the decay time of x is on the order of minutes [22] (Section 5.3). Bearing this fact in mind, we will investigate systems driven by SIN with faster decay time compared with that of x ($\gamma \gg 1$). Furthermore, in real-world systems, we expect that the $f(x)$ in Eq. (3) is often given by a complex nonlinear function and is also accompanied by non-trivial multiplicative noise expressed by an appropriate $g(x)$. These properties are different from those for financial engineering. In order to obtain the probability distribution $P(x,t)$, we use adiabatic elimination with eigenfunction expansion [23]. We obtain a higher-order Fokker–Planck equation (HFPE) with higher-order derivatives, not included in the conventional Fokker–Planck equation (FPE). We calculate stationary distributions of the systems described by the HFPE by using the perturbation expansion.

To investigate the effects of SIN, we consider stationary distributions for three cases: (a) parabolic and quartic bistable potentials with additive noise, (b) a quartic potential with multiplicative noise and (c) a stochastic gene expression model. In the additive noise case (a), we show that SIN changes the stationary distribution from exponential forms (the Boltzmann–Gibbs distribution) to non-exponential forms. At the same time, the stationary distributions of case (a) are sharpened because of noise-intensity fluctuations. In case (b), we show that the existence of noise-intensity fluctuations induces the transition of distributions: the stationary distributions are uni, bi, or trimodal, depending on the SIN parameters. It is important to note that the trimodal distribution does not emerge under white noise. In case (c) which we call the gene expression model, a nonlinear function is given as a drift term (change in expression levels). Thus, case (c) shows that our approximation scheme can be applied to general configurations that include non-trivial drift terms.

The remainder of this paper is organized as follows. In Section 2, we describe the model proposed in this paper. Adiabatic elimination with eigenfunction expansion is applied to the model in Section 3 (details of the derivation of the HFPE are explained in the Appendix). In Section 4, we proceed to the calculation of the stationary distributions of the obtained HFPE by using the perturbation expansion. In Section 5, we investigate effects of SIN in the three cases (a), (b), and (c) mentioned above. In Section 6, we analyze effects of higher-order derivatives in the HFPE on positivity and moments of distribution functions, and also consider the opposite case, in which a decay time of the noise-intensity fluctuations is very slow ($\gamma \to 0$). Finally, we give the conclusions in Section 7.