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Time to consensus

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ABSTRACT

The dynamics of a two-state decision-making model (DMM) with a tunable control parameter K is described. On all-to-all (ATA) networks, the system undergoes a phase transition at a critical value of $K_c = 1$. Scale-free networks were also found to undergo phase transitions, but the value of K_c increases as the scale-free exponent increases. Time to consensus is defined as a first passage time to a specified threshold of the global mean field, which represents a level of majority for the network. At criticality, the time to consensus for sparse networks with broad degree distributions (e.g., scale-free networks) approaches that of the ATA networks, although somewhat more cooperation (higher K_c) is required.

1. Introduction

Criticality is a property of complex systems that is well known in the physical sciences and has recently attracted the attention of an increasing number of researchers in other fields ranging from neurophysiology [1–3] to sociology [4,5]. In these latter fields, the concept of criticality is not yet clearly defined due to the lack of a rigorous theoretical approach that is experimentally verified. In spite of that shortcoming, we herein adopt the concept of criticality as a property of complex networks that undergo phase transitions and use renormalization group theory (as recommended by Gerhard Werner [6]) as the benchmark of this crucial concept.

Beggs and Timme [7] have recently emphasized that there exists some confusion about using the concept of criticality outside the physical sciences since it is usually explained using the Ising model. On the one hand, this physics-based model describes the influence of thermal fluctuations on an equilibrium process. On the other hand, neural criticality describes a non-equilibrium dynamic process. By the same token, the criticality concept used in sociology is connected to phase transitions [8,9] that generate complex (inverse power law) distributions of the sizes of clusters [4,8] with little or no attention paid to the temporal fluctuation associated with the emergence of that clustering [5]. Herein, we adopt the decision-making model (DMM) introduced by Bianco et al. [10] and further developed by Turalska et al. [11,12] that, although bearing some similarity to the Ising model [11], has dynamics generated by internal fluctuations rather than by the interaction with an external thermal reservoir. As we show herein, the time evolution of the DMM in the mean field approximation can be expressed through a nonlinear Langevin equation in the ideal case where each element in the network interacts with all the other elements, namely, in the ATA condition. However, there are good reasons [13] to believe that at and near criticality, the time evolution of the mean field is described by the same nonlinear Langevin equation even if the cooperation effort is stronger and coupling among the elements is not ATA.

There is great interest in the social sciences concerning the understanding of how agreement is reached by collections of cooperating individuals. If the level of cooperation is low, the group opinion tends to remain unchanged, even if the

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opinions of some individuals fluctuate. If the level of cooperation is high, however, the group opinion evolves quickly toward a state with a clear majority and tends to remain there for a long time. Between these extremes of no preference and global agreement is a critical level of cooperation in which large majorities can form, but these clusters are unstable and can potentially reverse quickly. We refer to a level of majority reached in the critical condition as "consensus". It is in fact this temporary consensus that enables a flock of birds to quickly change direction when a predator is sighted. The critical condition is typically characterized by a control parameter achieving a critical value. If the flock was in the supercritical state, with the control parameter above the critical value, the flock would not be able to react quickly enough to avoid the predator (and, of course, in the subcritical state with the control parameter below the critical value, the flock would not react at all to the predator!). This critical state is not only a useful description of the function of an "intelligent swarm" [14] but is also widely considered to be a necessary condition for functional connectivity in the brain [15].

The connection between the number of elements involved in reaching consensus and the time necessary to reach that consensus is a significant issue and has been widely discussed; see, for instance, Refs. [16–18]. Remarkably, the time to consensus with elements that are at the nodes of complex networks is much faster than for elements embedded in regular networks [16]. Moreover, the role of either committed minorities [17] or of voters in a latent state [18] makes the time to consensus proportional to the logarithm of the number of elements within the network rather than proportional to the network size itself. It is important to notice that the research in these papers is based on a game theory perspective, whereas the results presented herein arise from a different theoretical perspective based on the role of criticality [6].

In Section 2, we briefly review the DMM and in Section 3, we present a near mean field approximation in terms of a nonlinear Langevin equation. The steady state solution to the corresponding Fokker–Planck equation for the probability density allows us to estimate the time to reach consensus. In Section 5, it is argued that the time to consensus is a first passage time property of a complex network not unlike that frequently adopted in the literature to study, for instance, chemical reaction processes [19]. The traditional models in physical chemistry correspond to an excessively long time to reach consensus. In fact, the first passage time distribution has the typical Poisson structure:

$$\psi(t) = r\exp(-rt), \tag{1}$$

where the rate of transition out of the consensus state is given by

$$r = \omega_{coll}\cdot\exp\left(-\frac{Q}{D}\right), \tag{2}$$

where Q is the reaction barrier and D is a diffusion coefficient proportional to temperature in the theory of chemical reactions. The exponential is the Arrhenius factor derived through the assumption $Q \gg D$, which in this theory has the effect of creating an impressively long time scale separation between the fast molecular collisions, thereby making the time scale $1/r$ much larger than the molecular time scale $1/\omega_{coll}$. Herein we show that the distribution of times to consensus departs from the traditional Arrhenius-like perspective, creating a condition similar to that explored by the authors of Ref. [20] but strikingly different from the original chemical reaction theory of Kramers [21], where the chemical reaction process occurs over a time scale much longer than the one at which a canonical equilibrium is realized in the reactant well. We find that the time to consensus is much shorter than the time necessary to produce canonical equilibrium in these early physical models of cooperation. A second significant result is that to generate the fast transition to consensus it is not necessary to rest on the ATA condition where each element interacts with all the other elements. We evaluate a variety of sparse network topologies and find that networks with broad degree distributions [22] reach consensus nearly as fast as the ideal ATA network, but with far fewer links.

2. The decision-making model

Cooperative behavior occurs when elements in a network take action based in whole or in part on input received from other elements. From a network theory perspective, the elements can be considered interacting units linked by their ability to exert influence on other units. Here we consider the two-state DMM introduced by Bianco et al. [10] and developed by Turalska et al. [11,12] in which each element makes a decision to change its state based in part on input from its neighbors. This model is superficially similar to other binary models such as the voter model [23] and the majority rule model [24] but has the distinct advantage of including a tunable parameter K, the control parameter, that quantifies the degree of cooperation among the agents. This is particularly useful because it allows us to identify a critical value at which a phase transition occurs. Initially, the nodes of the network are randomly assigned values of 1 (state 1) or −1 (state 2) such that

$$\bar{\xi} \equiv \frac{1}{N}\sum_{i=1}^{N}\xi_i = 0, \tag{3}$$

where ξ_i is the state (1 or −1) of element i and N is the number of elements in the network. We "turn on" the cooperation at a time $t = 0$. At each subsequent time step, each element has the opportunity to change its state according to the transition rates:

$$g_{12} = g e^{K\left(\frac{M_2-M_1}{M}\right)} \quad \text{and} \quad g_{21} = g e^{-K\left(\frac{M_2-M_1}{M}\right)}, \tag{4}$$