Physica A 391 (2012) 3816–3827

Contents lists available at SciVerse ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

A study on the action of non-Gaussian noise on a Brownian particle

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ARTICLE INFO

Article history:
Received 15 August 2011
Received in revised form 6 February 2012
Available online 9 March 2012

Keywords:
Non Gaussian noise
Exact results
Time Averaging

ABSTRACT

We analytically address the non-equilibrium problem of a Brownian particle in contact with a thermal reservoir by means of a non-Gaussian Langevin noise term $\eta(t)$. The presence of noise kurtosis is akin to a second temperature reservoir acting on the system, and we exploit its consequences by means of studying a converging exact form for the stationary probability distribution.

1. Introduction

The Langevin equation is a useful tool for studying large classical systems where rapidly fluctuating forces are present [1]. These fluctuating forces are a consequence of the fast microscopic degrees of freedom of the system as initially explained by Einstein [2]. For most usual thermodynamic systems at equilibrium, these forces are quite well represented by white Gaussian noise, since they are the result of the coarse-graining of the actual quantum degrees of freedom [3]. Then, the Boltzmann–Gibbs (BG) distribution can be derived for general Hamiltonians [1,4–6].

However, for non usual noise (colored or not) the form of the stationary distribution is not BG and depends on the noise details [7–9], highlighting the essential role played by the properties of the Langevin force in the determination of the stationary state. In consequence, true equilibrium does not hold for general types of noise, but non-equilibrium stationary states might arise instead. In this case, characteristic functional approaches can be developed for non-Gaussian systems [10]. Many important systems fall into the category, such as is the case with Poisson noise, which possesses an infinite number of non-zero cumulants [3].

From a technological point of view, recent experiments show evidence of the presence of skewness in the distributions of currents for coherent quantum conductors induced by external magnetic fields [11–13]. This is a consequence of their far from equilibrium characteristics: electrical conductors stray away from the symmetries present at equilibrium and the associated equilibrium fluctuation relations are no longer valid [12]. These are hierarchies of fluctuation relations connecting the equilibrium value of the cumulants to linear and non-linear conductances and susceptibilities. They can be verified experimentally [13], confirming the non-Gaussian properties of these far from equilibrium settings. In fact, this is an example of a mechanism for building up cumulants of higher order than two. Similarly, for a thermal system far from equilibrium, the presence of kurtosis in the noise distribution shall induce the presence of kurtosis on the distribution of positions and velocities. Hence, we shall exploit some of the consequences of the presence of higher order cumulants on the noise distribution.

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0378-4371/$ – see front matter
doi:10.1016/j.physa.2012.03.012

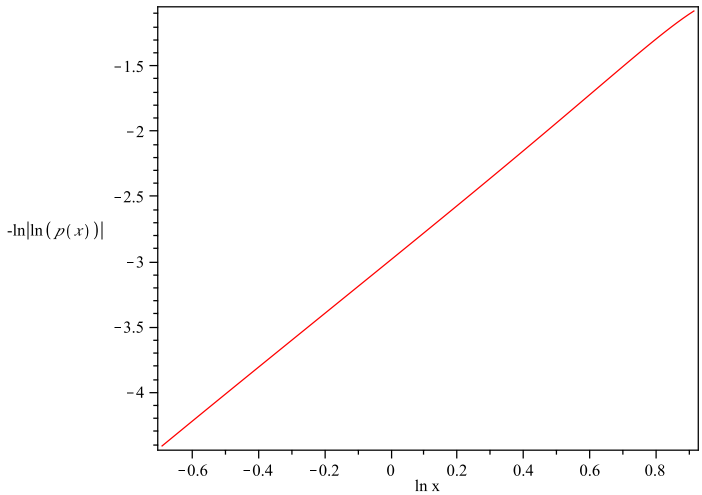

Fig. 1. In this plot, the position distribution is almost indistinguishable from a Gaussian for the low T_κ case. The parameter values are $k = 1.0$, $\gamma = 1.0$, $T = 0.5$, $T_\kappa = 1.5$, $m = 1.0$. This form of plotting the curve allows us to obtain the power for stretched exponentials as the angular coefficient of the curve. In this case it is 2.1, very close to a Gaussian, even for the case $T_\kappa/T = 3.0$.

We assume that there exists a non-equilibrium mechanism, breaking up the equilibrium symmetries and favoring the build up of higher order cumulant correlations in the bath, and that these non-Gaussian fluctuations will spill over the system (for instance, in Ref. [14] higher order cumulants arise for transferred particles, which corresponds to the heat for the present model). For simplicity sake, we choose a rather artificial form of symmetric noise which presents non-zero kurtosis, with small higher order cumulants since. According to Marcinkiewcz [15], the cumulant generating function cannot be a finite order polynomial, thus the kurtosis cannot be the highest non-zero cumulant and an infinite number of non-zero cumulants arise, such as for Poisson noise [16,17]. This type of distribution has fatter tails and smaller peaks than the normal distribution. We shall assume that the higher order cumulants are small and do not contribute significantly to the shape of the stationary distribution.

As we shall see, the presence of higher order noise cumulants (skewness and higher order ones) is equivalent to the coexistence of distinct temperatures, namely T and T_κ in the present case, respectively associated with the variance and the kurtosis of the noise. The equipartition theorem will not be affected, since T_κ will not arise from averages of quadratic energy terms. However, the presence of eventual non-quadratic energy terms would imply a feeding of energy into the system dependent on T and T_κ, as will be more clear later.

We will follow a method that is well suited to the obtaining of the exact stationary distribution for a linear Langevin or Langevin-like (non-Markovian) system [9,18], which has also been used to study, analytically, simple models for thermal conductance [19,20] and fluctuation theorems [21]. A recent article presents a treatment for Poisson noise, which itself presents infinite non-zero cumulants, and goes along similar lines to the current work [17]. However, in the present work we are able to obtain closed expressions for the probabilities, in terms of series expansions, instead of getting to know the cumulant generating function only [17]. The present model consists of a Brownian particle coupled to a harmonic potential, and a noise term with non-zero variance and kurtosis. The corresponding Langevin equation is then exactly solved via a time-averaging procedure. The main advantage of this method over Fokker–Planck approaches (FPE) is that, given the exact noise form (all of its cumulants), all the cumulants for the dynamical variables are calculated correctly, which can only be guaranteed for the first two cumulants in the case of the FPE. One of the interesting aspects of the present method is that we can deal quite easily with the presence of the mass of the Brownian particle, so all dynamical effects are taken into account.

Our stationary results can be expressed in terms of converging series, as can be seen in Fig. 1. For a range of values of the parameters the distributions can be numerically close to Gaussian ones. Another way of approaching this problem is by looking at the thermostat and the heat it injects into the system. This can also be obtained analytically and the results are in complete agreement with the stationary state ones, such as the equipartition theorem, which is still valid for the present model.

This paper is organized as follows. In Section 2, we present the model used. In Section 3, we develop the methodology for solving it via time-average. In Section 4, we discuss the results for the exact series for the distributions. In Section 5, we discuss the injection of energy into the system, and, in Section 6, we elaborate our conclusions.

2. Model

The kurtosis, defined commonly as $\gamma_2 \equiv \frac{\kappa_4}{\kappa_2^2}$, where κ_4 and κ_2 are the fourth and second order cumulants respectively, is a measure of the "bulging" of a distribution. The more peaked distributions will have higher kurtosis values (leptokurtic) whereas the broader ones will have smaller, or even negative, values for kurtosis (platykurtic). The kurtosis can also be