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On spurious and corrupted multifractality: The effects of additive noise, short-term memory and periodic trends

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ABSTRACT

We study the performance of multifractal detrended fluctuation analysis (MF-DFA) applied to long-term correlated and multifractal data records in the presence of additive white noise, short-term memory and periodicities. Such additions and disturbances that can be typically found in the observational records of various complex systems ranging from climate dynamics to physiology, network traffic, and finance. In monofractal records, we find that (i) additive white noise hardly results in spurious multifractality, but causes underestimated generalized Hurst exponents $h(q)$ for all q values; (ii) short-range correlations lead to pronounced crossovers in the generalized fluctuation functions $F_q(s)$ at positions that decrease with increasing moment q, thus causing significantly overestimated $h(q)$ for small q and spurious multifractality; (iii) periodicities like seasonal trends (with standard deviations comparable with the one of the studied process) result in spurious "reversed" multifractality where $h(q)$ increases with increasing q (except for very short time windows). We also show that in multifractal cascades moderate additions of noise, short-range memory, or periodic trends cause flawed results for $h(q)$ with $q < 2$, while $h(q)$ with $q > 2$ remains nearly unchanged.

1. Introduction

Many natural records characterizing the behavior of complex systems exhibit long-term persistence [1]. Prominent examples include physiological rhythms [2–7], traffic dynamics in telecommunication networks [8–10], as well as climatological [11–17] and financial [18–21] indicators. In these cases, the (linear) two-point autocorrelation function (ACF) $C_x(s)$ decays by a power law,

$$C_x(s) = \frac{1}{\sigma_x^2 (L-s)} \sum_{i=1}^{L-s} (x_i - \langle x \rangle)(x_{i+s} - \langle x \rangle) \sim s^{-\gamma}, \tag{1}$$

where σ_x denotes the standard deviation, $\langle x \rangle$ the mean, and γ the correlation exponent ($0 < \gamma < 1$) of the data set x_i, $i = 1, 2, \ldots, L$. Such correlations are named "long-term" since the mean correlation time $T_\times = \int_0^\infty C_x(s)\mathrm{d}s$ diverges in

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the limit of an infinitely long series [22]. For (linearly) uncorrelated x_i, $C_x(s) = 0$ for $s > 0$. If (linear) correlations exist up to a certain correlation time $s_\times$, then $C_x(s) > 0$ for $s < s_\times$ and $C_x(s) = 0$ for $s > s_\times$.

However, in many natural records a single scaling exponent is not sufficient for a full description of the correlation structure of the data set, but rather an infinite number of exponents is needed [22–35]. This kind of data is usually referred to as "multifractal", to distinguish from "monofractal" long-term correlated data characterized by a single scaling exponent. This happens, for example, when values of different magnitudes follow different scaling laws.

Several methods have been established to quantify multifractal scaling in non-stationary observational records. Among them, the wavelet transform modulus maxima (WTMM) method [36] and the multifractal detrended fluctuation analysis (MF-DFA) [37] are currently the most prominent and widely adopted methods, see [16,37,38] for comparisons. Both methods can eliminate additional polynomial trends in the data which make them superior to other methods like the structure function analysis [22] and the high-order autocorrelation functions [39]. Here we focus on MF-DFA [37] since it yields similar results as the WTMM but is considerably easier to implement.

In the MF-DFA one considers the profile, i.e., the cumulated data series $Y_j = \sum_{i=1}^{j}(x_i - \langle x \rangle)$, and splits it into L_s (non-overlapping) segments of size s. In each segment a local polynomial fit $y_\nu(j)$ of the profile of, e.g., second order is estimated. Then one determines the variance $F_\nu^2(s) = \frac{1}{s}\sum_{j=1}^{s}(Y_{[(\nu-1)s+j]} - y_\nu(j))^2$ between the local trend and the profile in each segment ν and determines a generalized fluctuation function $F_q(s)$,

$$F_q(s) \equiv \left\{ \frac{1}{L_s} \sum_{\nu=1}^{L_s} [F_\nu^2(s)]^{q/2} \right\}^{1/q}. \tag{2}$$

When $q = 0$, logarithmic averaging can be applied (for a detailed description, see Eq. (6) and discussions on p. 90 of [37]). In general, $F_q(s)$ scales with s as

$$F_q(s) \sim s^{h(q)} \tag{3}$$

with the generalized Hurst exponent $h(q)$. For a monofractal time series, $h(q)$ is independent of q and identical to the Hurst exponent H (see, e. g., [22]). For multifractal data, $h(q)$ depends on the chosen moment q. When the record is linearly long-term correlated (see Eq. (1)), $h(2) = 1 - \gamma/2$. In the absence of linear correlations (where $C_x(s) = 0$ for $s \geq 1$), $h(2) = 0.5$. In [37] it was shown that $h(q)$ is directly related to the scaling exponent $\tau(q)$ defined by the standard partition function-based multifractal formalism [25,39], via $\tau(q) = qh(q) - 1$, and is related to the multifractal spectrum $f(\alpha)$ via a Legendre transform $f(\alpha) = q[\alpha - h(q)] + 1$, where $\alpha = [\mathrm{d}\tau(q)/\mathrm{d}q]$.

Observational data often contain "artifacts" that make it difficult to detect and to quantify long-term correlations and possible multifractality. There are several types of such "artifacts".

1. Additive random noise. One prominent example can be found in physiological records, in particular in heartbeat intervals, where the random component is inherent in the measured process itself [40,41]. Here the noise component is informative and can be used to detect fibrillation [40]. Another possibility is the measurement noise that occurs due to the limited accuracy of the measurement equipment [42].

2. Short-term correlations. Prominent examples are temperature records [14,43,44] where on short time scales below 10 days there is a strong persistence which is superimposed to the long-range correlations.

3. Additional periodicities. Prominent examples can be found in climate records, where seasonal trend overlap long-term correlation effects. In mass service systems like computer and highway networks, periodicities originate from working and entertainment patterns (weekdays, holidays) and often result in several kinds of trends (daily, weekly, annual). For a recent study of seasonal trend impact on nonlinear memory in climate records, we refer to [45].

In this paper, we study how these artifacts: (a) when superimposed to long-term correlated record, give rise to spurious multifractality, and (b) when superimposed to a multifractal record, corrupt the multifractality.

2. Spurious multifractality in monofractal data sets

For generating long-term correlated data, we have used the Fourier-filtering technique, described, e.g., in [22,46], where the spectral coefficients of an uncorrelated random series are multiplied by $f^{-(1-\gamma)/2}$. The series, obtained by the inverse Fourier transform of these modified coefficients, exhibits power-law correlations on all time scales.

2.1. The impact of additive white noise

First, we consider the results of the multifractal analysis by MF-DFA in the presence of additive white noise. For corresponding detailed studies with normal DFA (just $q = 2$) we refer to [47]. The data sets are

$$y_n = x_n + Au_n, \tag{4}$$

where x_n is a long-term correlated series, u_n is a white noise series (both characterized by zero mean and unit variance), and A defines the noise level. Throughout the whole paper, we estimate $h(q)$ by fitting a power-law to $F_q(s)$ for each value of q by the least mean square technique with the scale range between $s = 10^3$ and 10^4 for 100 data sets of length $L = 2^{16}$.