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Non-stationary multifractality in stock returns

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HIGHLIGHTS

- Stock returns show significant time varying multifractal properties.
- New evidence of multifractality dependence on the unconditional return distribution.
- The heavy tails cannot explain fully the large multifractality fluctuations.

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ABSTRACT

We perform an extensive empirical analysis of scaling properties of equity returns, suggesting that financial data show time varying multifractal properties. This is obtained by comparing empirical observations of the weighted generalised Hurst exponent (wGHE) with time series simulated via Multifractal Random Walk (MRW) by Bacry et al. [E. Bacry, J. Delour, J.-F. Muzy, Multifractal random walk, Physical Review E 64 (2) (2001) 026103]. While dynamical wGHE computed on synthetic MRW series is consistent with a scenario where multifractality is constant over time, fluctuations in the dynamical wGHE observed in empirical data are not in agreement with a MRW with constant intermittency parameter. We test these hypotheses of constant multifractality considering different specifications of MRW model with fatter tails: in all cases considered, although the thickness of the tails accounts for most of the anomalous fluctuations of multifractality, it still cannot fully explain the observed fluctuations.

1. Introduction

The concept of multifractality in the context of finance has received much attention in the econophysics literature [1] over the last two decades. Many empirical studies have investigated financial data scaling behaviour [2–11] and several models have been proposed to account for the observed multifractal features [12–20]. Multifractal behaviour has become a downright stylised fact of financial market data [21,22], being observed across several classes of assets: from daily stock prices to foreign exchange rates and composite indices [6,3,23–25].

Multifractality is also particularly appealing for modelling financial markets as it offers a simple behavioural interpretation: looking at the volatility at different timescales is a very natural way to assess the impact of heterogeneous agents in the market and therefore any measure of scaling behaviour can convey information about the efficiency of a given market [22]. In this regard some authors have suggested, and confirmed through extensive empirical studies, that scaling exponents can be representative of the stage of development of a market [23,3]. The same works have convincingly shown that, through a hierarchy of scaling exponents, it is possible to classify markets according to their degree of development with emerging markets exhibiting scaling exponents significantly larger than those observed in developed markets.

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Let us recall that a process $X(t)$ with stationary increments is called multifractal if the following scaling law is observed

$$\mathbb{E}(|X(t+\tau)-X(t)|^q) \sim c_q \tau^{\zeta_q}, \tag{1}$$

where c_q is a constant and ζ_q is a non-linear function of q, called the scaling function. The departure of ζ_q from linearity is what distinguishes multifractal processes from uni-scaling processes and the degree of non-linearity of the scaling function can be accounted for by the *intermittency coefficient*, defined as $\lambda^2 = \zeta_0''$ [12]. Note that the last definition requires the scaling function to have a second derivative well defined in $q = 0$. (In practice, it is extremely difficult to estimate this coefficient from empirical measures.) For multifractal processes, the scaling in Eq. (1) holds for small τ, with $\tau/T \ll 1$, where T is some larger scale called the *integral scale* [12]. This means that if the integral scale of the process is not large enough compared to the resolution of the increments, although the scaling may in principle hold, the definition does not hold any longer.

In order to estimate the scaling function from real data one must resort to the scaling of the empirical moments

$$\frac{1}{N-\tau+1}\sum_{t=0}^{N-\tau}|r_{t,\tau}|^q \sim \tau^{\zeta_q^*}, \tag{2}$$

where we denote $r_{t,\tau} = \log p_{t+\tau} - \log p_t$ the log-return at time t and scale τ, with p_t the asset price at time t, ζ_q^* the empirical scaling function and N the length of the time series. It is well known [26,27] that the empirically estimated ζ_q^* is significant only for small values of q and one therefore needs to be careful in interpreting the scaling beyond a certain q. When the scaling (2) is observed, one can define the generalised Hurst exponent $H(q)$ (GHE) via

$$\zeta_q^* = qH(q). \tag{3}$$

The exponent $H(q)$ is non-linear in q for multi-scaling processes, whereas it reduces to a constant H if the process is uni-scaling.

The time evolution of the scaling function, measured via the GHE, can be useful to track time varying properties of the market. For this reason ζ_q^* has also been studied dynamically via the time dependent (or local) Hurst exponent [28–33]. In a recent publication [34] the authors have observed large fluctuations in multifractality measured via the GHE in empirical daily data across different stock sectors. Specifically, the authors considered as a measure of multifractal behaviour the quantity

$$\Delta H^w(q,q') = H^w(q) - H^w(q'), \quad q \neq q', \tag{4}$$

where $H^w(q)$ is the weighted generalised Hurst exponent (wGHE) [34]. The weighting procedure incrementally damps effects from past return fluctuations. The dynamical evolution of $\Delta H^w(q,q')$ (which we shall label $\Delta H_t^w(q,q')$) is useful to track changes in multifractality occurring over time.

When facing the task of ascertaining the nature of dynamical fluctuations in these quantities, the subtle issue is being able to distinguish between spurious statistical fluctuations, which are due to the finiteness of the sample and noise, and true structural changes in the underlying multifractal process. In this paper we study the problem of validating dynamical fluctuations of the scaling functions, performing an empirical analysis of stock returns and comparing their properties with those of synthetic multifractal series. Among all existing models we focus on the Multifractal Random Walk (MRW) introduced by Bacry et al. [12] because of its parsimonious formulation and its success in the econophysics literature.

This paper is organised as follows. In Section 2 we review the main properties of the MRW and establish the connection with the generalised Hurst exponent. In Section 3, after introducing the statistical testing procedure, we report the main findings on the varying multifractality of empirical stock returns data. A summary and conclusive remarks are drawn in Section 4.

2. Generalities on multifractal random walk

Accordingly with Bacry et al. [12], the multifractal random walk (MRW) can be viewed as a stochastic volatility model constructed by taking the limit for $\Delta t \to 0$ of the process

$$X_{\Delta t}(t) = \sum_{k=1}^{t/\Delta t} \epsilon_{\Delta t}(k)\, \mathrm{e}^{\omega_{\Delta t}(k)} \tag{5}$$

with $\epsilon_{\Delta t}(k)$ a Gaussian white noise with variance $\sigma^2\Delta t$ and $\mathrm{e}^{\omega_{\Delta t}(k)}$ a stochastic volatility uncorrelated with ϵ. By taking $\omega_{\Delta t}(k)$ as a stationary Gaussian process, we have log-normal volatility components. What distinguishes the limit $\Delta t \to 0$ of $X_{\Delta t}(t)$ from a Brownian motion is the choice of the auto covariance structure of the process $\omega_{\Delta t}(k)$, which is chosen, according to cascade-like processes [35], as

$$\mathrm{Cov}(\omega_{\Delta t}(k), \omega_{\Delta t}(k+h)) = \begin{cases} \lambda^2 \log \dfrac{T}{(1+h)\Delta t}, & h \le T/\Delta t - 1 \\ 0, \quad \text{otherwise.} \end{cases} \tag{6}$$