Physica A xx (xxxx) xxx–xxx

Contents lists available at SciVerse ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Investigation of classical and fractional Bose–Einstein condensation for harmonic potential

Q1 Neslihan Üzar [a,*], Sedat Ballıkaya [b]

[a] *Solid State Physics Division, Physics Department, University of Istanbul, 34134-Vezneciler, Istanbul, Turkey*

[b] *General Physics Division, Physics Department, University of Istanbul, 34134-Vezneciler, Istanbul, Turkey*

ARTICLE INFO

Article history:
Received 14 September 2012
Available online xxxx

Keywords:
Bose–Einstein condensation
Gross–Pitaevskii equation
Homotopy perturbation method
Caputo fractional derivative operator

ABSTRACT

In this study, classical and fractional Gross–Pitaevskii (GP) equations were solved for harmonic potential and repulsive interactions between the boson particles using the Homotopy Perturbation Method (HPM) to investigate the ground state dynamics of Bose–Einstein Condensation (BEC). The purpose of writing fractional GP equations is to consider the system in a more realistic manner. The memory effects of non-Markovian processes involving long-range interactions between bosons with the restriction of the ergodic hypothesis and the effect of non-Gaussian distributions of bosons in the condensation can be taken into account with time fractional and space fractional GP equations, respectively. The obtained results of the fractional GP equations differ from the results of the classical one. While the Gauss distribution describing the homogeneous, reversible and unitary system is obtained from the classical GP equation, the probability density of the solution function of fractional GP equations is non-conserved. This situation describes the inhomogeneous, irreversible and non-unitary systems.

1. Introduction

BEC was firstly identified by Einstein and Bose theoretically in 1924 [1,2], and observed experimentally with alkali metals such as rubidium (Rb) and sodium (Na) atoms in 1995 [3,4]. After that the physical reality of BEC was well understood and the interest on this topic has increased significantly. BEC is a very important and topical subject due to the explanation of quantum effects seen on a macroscopic scale [5,6], the nature of phase transitions [7,8], transmission of matter [9] and the behavior of superconductivity and superfluids [10,11]. In this respect, not only experimental studies are important but theoretical studies too. For the experimental investigations of BEC, bosons should be cooled to nano-Kelvin temperatures and trapped in a specific region of space. The laser evaporating, cooling and magneto-optical techniques are commonly used for the cooling processes of bosons [3,4,12,13]. A combination of these three techniques allows the system to achieve nano-Kelvin temperatures. The first experiment on BEC was made with ^{87}Rb atoms [3]. These atoms were cooled in a magnetic field with the laser evaporating process. A magnetic field was provided to confine the boson atoms. Anderson et al. precooled the ^{87}Rb atoms optically. Then, they trapped the bosons in a combination of a quadrupole magnetic field and a uniform transverse field which was rotated at 7.5 kHz [3]. This mechanism created an axially symmetric harmonic potential. In other words, this magnetically created potential can be modeled with a harmonic potential as a good approximation. Therefore, most of the studies on BEC were done with a harmonic potential. There are also some studies on thermodynamic properties of BEC for different potentials such as quartic potential, anisotropic potential and so on [14,15]. The external potential does not only keep the atoms together, the macroscopic behavior of BEC matter and interactions between the bosons also depend on this potential.

* Corresponding author. Tel.: +90 535 427 12 71.
E-mail addresses: neslihanuzar@gmail.com, neslihanuzar@yahoo.com (N. Üzar).

0378-4371/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.physa.2012.11.039

Condensation of weakly interacting boson atoms is described with the classical GP equation at $T = 0$ K. The GP equation, which has been derived by the use of mean-field theory, is a non-linear Schrödinger equation [16,17]. The classical GP equation defines BEC with a single macroscopic wave packet. The interactions between the boson particles are in the form of binary scattering and the media where this phenomenon takes place is homogeneous.

An open system always interacts with its environment and there are long-range interactions between the particles. Also, the system's temperature cannot be 0 K at any time. These lead to some events such as dissipation and decoherence due to irreversible transitions of information from the system to the environment or vice versa and inhomogeneous condensation. The first case is known as the memory effect of the non-Markovian process. The second yields a non-Gaussian distribution. Markovian and non-Markovian processes define stochastic processes. A Markovian process predicts the future state of a system based solely on its present state. So, the future and the past of the system are independent, whereas a non-Markovian process takes into account the whole past history of the system of interest. Additionally, non-conservative systems with non-Gaussian distribution describe the irreversible systems. The theoretical and experimental studies on BEC indicate inconsistency such as critical temperature [3,4,6,18]. This problem may occur from the above-mentioned factors.

In recent years, fractional integrals and derivatives in the calculation methods have been used for the explanation of physical phenomena which do not comply with the laws of classical statistical physics [19,20]. Space-fractional and time-fractional differo-integral operators are used to take into account the fractional nature of space where physical events take place and the memory effect of the non-Markovian process of the system, respectively [21–23]. We can clearly say that the classical GP equation serves the idealization of a bosonic system. So it is insufficient to investigate the physical events at the ground-state energy level where the condensation takes place. Therefore fractional GP equations can be used to describe the real bosonic system.

In this paper, our aim is to study the fractional GP equations by investigating the effects of some physical events on BEC. The Caputo fractional derivative operator is used for fractional GP equations. Classical and fractional GP equations will be separately solved using HPM and the obtained results will be compared. Firstly, the classical and fractional GP equations are defined. Later some general information is given for using in the solution phase.

2. Time dependent classical and fractional GP equations

The classical GP equation can explain the quantum effects in BEC with the macroscopic ground state wave function $\Psi_0(x,t)$. This wave function, which evaluates with time and space, describes BEC as a single particle [16,17]. Thereby, the system is simplified. The 1-D classical GP equation is given as

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi_0(x,t) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_{\text{ext}}(x) + g\,|\Psi_0(x,t)|^2\right)\Psi_0(x,t) \tag{1}$$

where $V_{\text{ext}}(x)$ is the external potential for trapping the boson particles. Because atoms are trapped in specific regions of space, they cannot disperse everywhere in space and this event helps the condensation. $g = 4\pi\,\hbar^2\,a_s/m$ is known as the coupling constant. It defines the interaction types between the bosons. Here a_s is the interaction parameter. $a_s < 0$ and $a_s > 0$ represent the attractive and repulsive interaction between the bosons, respectively. m is the mass of a boson particle.

Fractional GP equations are built to investigate boson systems in a more realistic way. One of them is the time-fractional GP equation. The memory effect, long-range interaction and restriction of entropic and ergodic hypotheses [24] in BEC can be taken into account with the time-fractional GP equation. It may be defined as

$$\mathrm{i}\,\hbar\,{}_0D_t^{\alpha}\,\psi(x,t) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_{\text{ext}}(x) + g\,|\psi(x,t)|^2\right)\psi(x,t) \quad 0 < \alpha \le 1 \tag{2}$$

where ${}_0D_t^{\alpha}$ is the Caputo fractional derivative operator. In the literature mostly the Caputo fractional derivative operator is used, because the use of the Caputo fractional operator in the fractional-order differential equations allows us to take the initial condition in the same form used in the integer-order differential equation. Also, the Caputo definition is more appropriate for making physical interpretations. Therefore, it is preferred in the physical problems and there are some studies in the literature, such as applications to the Schrödinger equation and Klein–Gordon equation [21,25].

Similarly, the non-Gaussian distribution of bosons in BEC and the fractal nature of space can be written in the form

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(x,t) = \left(-\frac{\hbar^2}{2m}\,{}_0D_x^{\beta} + V_{\text{ext}}(x) + g\,|\psi(x,t)|^2\right)\psi(x,t) \quad 0 < \beta \le 2 \tag{3}$$

where α and β refer to the fractional order of derivative of time and space, respectively.

According to the definitions given above, the 1-D time and space fractional GP equation can be denoted as the following

$$\mathrm{i}\,\hbar\,{}_0D_t^{\alpha}\,\psi(x,t) = \left(-\frac{\hbar^2}{2m}\,{}_0D_x^{\beta} + V_{\text{ext}}(x) + g\,|\psi(x,t)|^2\right)\psi(x,t). \tag{4}$$

Before solving the classical and fractional GP equations, some basic rules for fractional calculus including the Caputo fractional derivative operator are given as follow.

Please cite this article in press as: N. Üzar, S. Ballıkaya, Investigation of classical and fractional Bose–Einstein condensation for harmonic potential, Physica A (2012), doi:10.1016/j.physa.2012.11.039