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Arithmetic Brownian motion subordinated by tempered stable and inverse tempered stable processes

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ABSTRACT

In the last decade the subordinated processes have become popular and have found many practical applications. Therefore in this paper we examine two processes related to time-changed (subordinated) classical Brownian motion with drift (called arithmetic Brownian motion). The first one, so called normal tempered stable, is related to the tempered stable subordinator, while the second one – to the inverse tempered stable process. We compare the main properties (such as probability density functions, Laplace transforms, ensemble averaged mean squared displacements) of such two subordinated processes and propose the parameters' estimation procedures. Moreover we calibrate the analyzed systems to real data related to indoor air quality.

1. Introduction

Processes based on the Brownian motion were considered in many aspects and have found various practical applications [1–8]. But the assumption of normality for the observations seems not to be reasonable in the number of examined phenomenon. Therefore in many Gaussian models the Brownian motion is replaced by its various modifications. One of the simplest modification is the extension of Gaussian by another distribution, for example the α-stable one. Processes based on the stable distribution are very useful in modeling data that exhibit fat tails. For example, the classical Ornstein–Uhlenbeck process was extended to the stable case and analyzed in Refs. [9,10] as a suitable model for financial data description, see also Ref. [11]. Another possibility of modification for Brownian-type processes is the introduction of time-changed Brownian models. This extension is related to replacement of real time in Brownian systems by a non-decreasing Lévy process (called a subordinator), that in this case plays a role of random (operational) time. The new process is called subordinate. The idea of subordination was introduced in 1949 by Bochner [12] and expounded in his book in Ref. [13]. The theory of subordinated processes is also explored in detail in Ref. [14]. The subordinated processes were studied in many areas of interest, for example in finance [15–18], physics [19–22], ecology [23] and biology [24].

The subordinated processes based on the diffusive Brownian motion were considered in many disciplines. The Brownian motion subordinated by the gamma process, so called variance gamma, is analyzed in Ref. [25] in the context of option price modeling. The other applications and main characteristics of such system can be found in Ref. [26]. Subordination of Brownian motion by the inverse Gaussian process is called the normal inverse Gaussian (NIG) and was considered for example in Ref. [27] and proposed for modeling turbulence and financial data. Applications of NIG processes to asset returns are also shown in Ref. [28] while analysis of real environmental data by using NIG distribution is presented in Ref. [29]. Let us mention that the Brownian motion driven by a strictly increasing Lévy subordinator is also a Lévy process, moreover when the subordinator is a temporally homogeneous Markov process then the subordinate has also this property, [14].

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Another possibility of subordination is the replacement of real time in Brownian diffusion by inverse subordinators, and processes that arise after this transformation exhibit properties of anomalous diffusion. In the domain of anomalous diffusion the typical approach is based on continuous time random walk (CTRW), [30,31], and the subordinated Lévy process can be treated as a limit in the distribution of CTRW, [32]. The key issue in the framework of CTRW as well as in the subordination technique is the waiting-times distribution corresponding to observed constant time periods [33]. In the last decade the anomalous diffusion processes were analyzed by a various number of authors in many disciplines. For example the subordinated Brownian motion driven by an inverse α-stable subordinator was considered in Refs. [11,32,34,35], the inverse tempered stable subordinator was examined in Refs. [33,36–38], while the inverse gamma process was mentioned in Ref. [39]. The general case of Lévy processes that can play the role of inverse subordinators were explored for example in Refs. [40–42]. In general, the anomalous diffusion models are used for systems that exhibit properties not observable in diffusive processes, such as nonlinear in time mean squared displacement or visible constant time periods.

In this paper we examine two processes related to subordinated Brownian motion. The first one, so called normal tempered stable [43,44], is a Brownian motion with drift (called arithmetic Brownian motion, ABM) driven by a tempered stable subordinator, while the second one is an ABM subordinated by an inverse tempered stable subordinator. The tempered stable processes are extensions of the α-stable Lévy systems but possess also the properties of Gaussian models, therefore in the last few years they have become popular and very useful in the description of many real data, [10,44,45]. We compare the main statistical properties of the ABM driven by a tempered stable and an inverse tempered stable subordinator. Moreover in two considered cases we propose the parameters' estimation procedures and validate them. In order to illustrate the theoretical results we calibrate the examined processes to real data related to indoor air quality.

The rest of the paper is organized as follows: In Section 2 we introduce the ABM and tempered stable subordinator. We present the main properties of those processes and define the time-changed ABM driven by the tempered stable process. For this system we also examine the main statistical characteristics, such as the probability density function, Laplace transform and ensemble averaged mean squared displacement that can be an useful tool for distinction between diffusion and anomalous diffusion models. In this section we propose also the estimation procedure based on the distance between theoretical and empirical Laplace transforms and validate it. In Section 3 we examine the inverse tempered stable subordinator and its main statistical properties as well as define the ABM driven by the inverse tempered stable process. In this section we also present the estimation procedure for unknown parameters. In Section 4 we model real data sets related to indoor air quality using the mentioned subordinated processes. The last section contains conclusions.

2. Arithmetic Brownian motion with tempered stable subordinator

2.1. Arithmetic Brownian motion

The arithmetic Brownian motion (ABM) is a process $\{X(t),\ t \geq 0\}$ defined by Refs. [33,46]:

$$\mathrm{d}X(t) = \beta \mathrm{d}t + \mathrm{d}B(t), \tag{1}$$

where $\{B(t),\ t \geq 0\}$ is a classical Brownian motion. The solution of Eq. (1) takes the form

$$X(t) = X(0) + \beta t + B(t). \tag{2}$$

In the further analysis we assume $X(0) = 0$ with probability one and in this case the process defined above has Gaussian distribution with mean βt and variance t. It is called also Brownian motion with drift, [33] and first of all was used to description of stock prices [1,47]. It is an extension of the classical Brownian motion therefore its modifications have found many other practical applications, like diffusion in liquids modeling, [48] and description of hydrology time series [49].

2.2. Tempered stable subordinator

The tempered stable subordinator $\{T(t),\ t \geq 0\}$ is a strictly increasing Lévy process with tempered stable increments, i.e. with the following Laplace transform, [37,38]:

$$\langle \mathrm{e}^{-zT(t)} \rangle = \mathrm{e}^{t(\lambda^{\alpha} - (\lambda+z)^{\alpha})}, \quad \lambda > 0,\ 0 < \alpha < 1. \tag{3}$$

When $\lambda = 0$, then $\{T(t)\}$ becomes totally skewed α-stable Lévy process. The probability density function (pdf) of tempered stable subordinator can be expressed in the following form:

$$f_{T(t)}(x) = \mathrm{e}^{-\lambda x + \lambda^{\alpha} t} f_{U(t)}(x), \tag{4}$$

where $f_{U(t)}(\cdot)$ is a pdf of the process $\{U(t),\ t \geq 0\}$—a totally skewed α-stable Lévy motion with the stability index α, [33,50]. Using tail approximation of stable density, [51], we obtain the following:

$$f_{T(t)}(x) \sim 2\alpha c_{\alpha} \mathrm{e}^{-\lambda x + \lambda^{\alpha} t} t^{\alpha} x^{-(\alpha+1)}, \quad x \to \infty \tag{5}$$