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Network formation under linking constraints ☆

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H I G H L I G H T S

- To the best of our knowledge this is the first paper that studies the impact of linking constraints on network formation.
- We characterize stable and strictly stable architectures under linking constraints.
- We study Bala and Goyal's dynamic model under linking constraints.

A R T I C L E   I N F O

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We study the effects of linking constraints on stability, efficiency and network formation. An exogenous "link-constraining system" specifies the admissible links. It is assumed that each player may initiate links only with players within a specified set of players, thus restricting the feasible strategies and networks. In this setting, we examine the impact of such constraints on stable/efficient architectures and on dynamics.

1. Introduction

In recent years, the study of the economics of networks has attracted considerable attention from researchers and become one of the hottest topics of economic research.[2] The economics of networks is, in Goyal's words, "*an ambitious research program which combines aspects of markets (e.g., prices and competition) along with explicit patterns of connections between individual entities to explain economic phenomena*" [1, p. 6].

Several seminal papers provide the basic models of strategic formation of networks. In the simplest model, links are formed unilaterally [4,5]. In this setting, Bala and Goyal [6] study Nash and strict Nash stability and provide a dynamic model. A model where links are formed on the basis of bilateral agreements is studied by Jackson and Wolinsky [7], who introduce the notion of pairwise stability. These seminal papers assume homogeneity across players and that the current network is common knowledge to all node-players. These models have been extended in different directions. Bala and Goyal [8]

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1 http://www.bridgebilbao.es.

2 Some recent books surveying this literature are Goyal [1], Jackson [2] and Vega-Redondo [3].

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introduce imperfect reliability of links. Galeotti et al. [9] consider heterogeneous players, while Bloch and Dutta [10] consider endogenous link strength. The common knowledge assumption may be unrealistic in many cases, and indeed is dropped by McBride [11], who studies the effects of limited perception, namely, assuming that each node-player perceives the current network only up to a certain distance from the node.

In the seminal models, networks provide a means for the flow of information or other benefits through the links, but the current network is assumed to be common knowledge to all players, who may unrestrictedly initiate links with any other players. This may be an unrealistic assumption in some cases and, in general, the larger the number of agents and the network is, the more unrealistic it will be. Due to what can generically be referred to as "institutional constraints" (social, cultural, linguistic, geographical, economic, etc.), individuals may often see only "part of the world" and initiate links only within that part or a part of that part. Thus, it seems more realistic to assume that each individual may initiate links only with a subset of players. In a way, this is an unorthodox approach if, as put by Goyal, "*the theoretical research on network effects* (…) *is motivated by the idea that*, within the same group (in italics), *individuals will have different connections and that this difference in connections will have a bearing on their behavior*" [1, p. 7]. Nevertheless, this is the approach adopted here and it is worth remarking that the no-constraints assumption is in fact a particular case of the more general setting adopted here. In particular, this allows Bala and Goyal's [6] "two-way flow" basic model,[3] on which we concentrate in this paper, to be integrated into a wider model which sheds new light on various conclusions of their model, showing which prevail and up to which point, and which do not in this wider setting.

Based on this idea, this paper focuses on the effects of such institutional constraints on stability, efficiency and network formation, assuming that an exogenous "link-constraining system" specifies which links are feasible, thus restricting the feasible networks. Such a link-constraining system can be specified by an underlying undirected network consisting of the admissible links or equivalently by a collection of sets specifying each player's "reach", i.e., the set of players with whom she may initiate links. It is also assumed that players in the same component of the link-constraining underlying network have common knowledge of the part of the current network connecting individuals in that component. In other words, players know the part of the current network that concerns their payoff, given that only players in the same component of the underlying network may influence their payoff. Further note that this model collapses to Bala and Goyal's [6] unrestricted setting for the particular case in which the underling constraining network is the complete network.[4]

For any given link-constraining system, we only pay attention to the admissible networks (i.e., those consistent with it) and first extend Bala and Goyal's [6] notion of a Nash network as those admissible networks where no player has an incentive to change her strategy, i.e., her choice of admissible links. We then easily extend their characterization of Nash networks as those among the admissible networks which are minimally connected. The set of such Nash networks is thus a subset of the set of Bala and Goyal's unrestricted Nash networks. Then the notion of a strict Nash network is also naturally extended to this setting. Now a strict Nash network is a network consistent with the link-constraining system where no player may initiate and/or delete any admissible link(s) without loss. By contrast with Nash networks, things turn out to be more complicated with *strict* Nash networks. In Bala and Goyal's setting, the center-sponsored star is the only (non-empty) architecture of strict Nash networks, while in our setting the center-sponsored star architecture is feasible only when there is at least one player that can initiate a link with any other player. Nevertheless, even when the center-sponsored star architecture is feasible, this might not be the only possible architecture of strict Nash networks. A variety of architectures of strict Nash networks appear when constraints are introduced. Nevertheless, some patterns are common to these architectures. Moreover, a full characterization of all strict Nash networks for a link-constraining system is provided by means of a condition that encapsulates synthetically the essence of the architecture of these networks, embodying a clear hierarchical principle. The main features of their architectures, where stars continue to play a prominent role, are studied. Particular attention is paid to the role of certain players who are the unique players in the intersection of other players' reach, by means of whom different groups can be connected. It turns out that the two-way flow model under constraints yields as strict Nash networks the paradigm of hierarchical structures: either oriented diverging trees (also called "arborescences" in graph theory) or a sort of "grafted" overlapping oriented trees. The latter are proved to be possible only when there are "hinge-players", i.e., players who are the *unique* node in the intersection of other players' reach.

We then apply Bala and Goyal's dynamic model, where starting from any initial network each player with some positive probability plays a best response or randomizes across them when there is more than one, otherwise the player exhibits inertia, i.e., keeps her links unchanged. In this way, a Markov chain on the state space of all networks is defined. In Bala and Goyal's setting, the absorbing states are precisely the strict Nash networks and they prove that starting from any network the dynamic process converges to a strict Nash network (i.e., the empty network or a center-sponsored star) with probability 1. When adapted to our setting the best response dynamic model *does not* necessarily lead to strict Nash networks. The reason is that in our more complex setting this dynamic process may lead to the formation of partially stable "incomplete" strict

[3] In this model players may initiate links unilaterally, information flows through links in both directions and a player's payoff is increasing with the number of players she is connected with directly or indirectly.

[4] A different and in a sense opposite form of link-formation constraint is studied in Ref. [12], where link formation takes place around a pre-existing core network. In our setting the feasible networks are those *contained* in the underlying one, while in Haller's model the feasible networks are those *containing* the pre-existing one. Thus, in our case the complete network yields the benchmark model, while in Haller's this is so for the empty core network. As in Haller's model the pre-existing network is not a set of forbidden links, but a set of publicly provided links; there is no further relationship with the model considered here.