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The quasi-periodicity of the minority game revisited

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HIGHLIGHTS

- We study the sequence of minority sides within a graph-theoretical framework.
- We define a choosing rule that facilitates the understanding of the MG sequences.
- We prove some results applicable to sequences which meet periodicity and PTD.

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ABSTRACT

We analyze two well-known related aspects regarding the sequence of minority sides from the Minority Game (*MG*) in its symmetric phase: period-two dynamics and quasi-periodic behavior. We also study the sequence of minority sides in a general way within a graph-theoretical framework. In order to analyze the outcome dynamics of the MG, it is useful to define the MG^{prior}, namely an *MG* with a new *choosing rule* of the strategy to play, which takes into account both prior preferences and game information. In this way, each time an agent is undecided because two of her best strategies predict different choices while being equally successful so far, she selects her *a priori favorite* strategy to play, instead of performing a random tie-break as in the *MG*. This new choosing rule leaves the generic behavior of the model unaffected and simplifies the game analysis. Furthermore, interesting properties arise which are only partially present in the *MG*, like the *quasi-periodic* behavior of the sequence of minority sides, which turns out to be *periodic* for the MG^{prior}.

1. Introduction

The Minority Game (*MG*) [1] is an agent-based model inspired in real complex systems which presents interesting collective properties like coordination among agents [2–4]. In the original formulation of the game, N agents (usually odd) must simultaneously choose one out of two alternatives: 0 or 1, and the winners are those who happen to be in the minority group. The *MG* tries to capture essential characteristics of some real situations in which belonging to the minority group turns out to be the most convenient situation (as, e.g., financial systems, traffic problems, and data networks).

At each step t of the game, $N_0(t)$ agents choose side 0 and $N_1(t)$ agents choose side 1, so that $N_0(t) + N_1(t) = N$. The system *state* $\mu \in \{0, 1\}^m$ is the only global information available for agents to make decisions. After each step, the state μ is updated on the basis of a certain *updating rule*. For example, in the original MG, μ is determined by the sequence of minority sides in the last m steps of the game.

A *strategy* is a function that assigns a prediction (0 or 1) to each possible state. In our case, every agent has $s = 2$ strategies at hand and at each step of the game, she plays using her best one. Whenever an agent's two strategies are equally ranked, she chooses one of them randomly.

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An *instance I* of the *MG* with $s = 2$ is a particular assignment of two strategies to the agents, $I = \{(e_1^1, e_2^1), (e_1^2, e_2^2), \ldots\}$. For $i = 1, \ldots, N$, the pair (e_1^i, e_2^i) represents the set of strategies assigned to the agent *i*. We define a *realization* $\mathcal{E}$ of the game as a pair $\mathcal{E} = \{\mathcal{S}_{\mathcal{E}}, I\}$, where $\mathcal{S}_{\mathcal{E}} = \{\tilde{\mu}^1, \tilde{\mu}^2, \ldots\}$ is a sequence of states (generated by any updating rule), and I is an instance of the *MG*.

The most studied variable in the *MG* is the reduced variance $\sigma^2/N = \langle (N_1 - N/2)^2/N \rangle_{\mathcal{E}}$ [5]. It measures the population's waste of resources by averaging – over time and over different realizations $\mathcal{E}$ – the quadratic deviation from $N/2$ of the number of agents that choose a fixed side (for example, N_1). When crowds emerge in the game, their contribution to σ^2/N is very important, indicating that fewer resources are being allocated to the population as a whole. On the other hand, for certain values of the parameters m, N, and s, σ^2/N results smaller than that obtained for a game in which each of the N agents randomly chooses between the two sides.

Period Two Dynamics (PTD) in the sequence of the minority sides was first observed by Savit et al. [6] within the symmetric phase of the MG. The PTD can be summarized in the following way: if a state μ appears at step t and this appearance is odd (i.e., the first, third, etc. step in which state μ occurs), then in the next (and hence even) appearance of μ, the outcome of the game is *very likely* to be the opposite of that obtained in the step t. Broadly speaking, this dynamics is due to the fact that on even occurrences of μ, crowds of agents will move together to the side rewarded in the previous odd appearance of the same state [6]. When PTD is met with probability 1 (i.e., when $P_{PTD} = 1$), we call it *Strict Period Two Dynamics (SPTD)*.

Acosta et al. have analytically solved the Full Strategy Minority Game (*FSMG*) [7,8], a maximal instance of the *MG*, which includes a single copy of every potential agent. For example, in a game with parameters m and $s = 2$, the number of *potential* agents is $\mathcal{N} = \binom{\mathcal{L}}{2} + \mathcal{L}$, where the first term represents all agents with two different strategies and the second term represents the number of agents whose two strategies are identical. Thus, the number of agents of the *FSMG* is a function of m. Certain strategies' symmetries, broken in the *MG*, can be fully exploited in the *FSMG*. This approach leads us to show that the *FSMG* verifies the SPTD for even occurrences of the states. The advantage of this approach lies in considering the *MG* as a statistical sample of size N of the *FSMG*. As a consequence, theoretical results for the *FSMG* can be used to compute approximated values of the key variable σ^2/N for the standard *MG* in the symmetric phase, as well as for other versions of the *MG* based on different updating rules, like the random updating rule introduced in Ref. [9] (MG_{rand}) and the periodic updating rule introduced in Ref. [10] (denoted by MG_{per}).

It is important to note that by an *even (resp. odd) occurrence* of a state we refer to the situation in which the current state μ_p at step t has appeared an odd (resp. even) number of times up to the step $t - 1$. Since μ_p appears again at step t, we consider this appearance to be an even (resp. odd) occurrence. This is an important remark in order to understand the results in Refs. [7,8] and the Appendix of this paper.

In this work, we first show that PTD is also met for odd occurrences of the states in the symmetric phase, but this is not accompanied by a crowd effect. This dynamics is reported as the antipersistence of the attendance. Jefferies et al. [11] show the MG behaves as a stochastically-disturbed deterministic system due to the random rule to resolve situations of tied strategies, by averaging over this stochasticity in order to get a deterministic dynamics of the MG. They also show that the trajectory of the outcomes of the deterministic MG on a de Bruijn graph is periodic within an Eulerian trail. Our article is closely related with that of Jefferies et al. Indeed, we use the FSMG framework instead of the RSS (Reduced Strategy Space), and we take into account the effect of undecided agents, which in fact not change the underlying dynamics. We also give elementary proofs for the connection among the periodicity of the MG outcomes, the Eulerian path on the de Bruijn graph, and the antipersistence of the attendance (or the PTD). On the other hand, the works of Zheng et al. [12] and Liaw et al. [10] show the existence of quasiperiods in the sequence of minority sides in the *MG*. In fact, the outcome of minority sides resulting from a game with $m = 2$, we can observe that there are two particular sequences of size 8 which appear several times in the sequence (though both sequences do not necessarily appear in the same realization of the game). These facts encourage us to propose an alternative *choosing rule* of the strategy to play, for which each agent has an *a priori* favorite strategy to use in case of indecision. We call MG^{prior} the resulting model. This modification holds the same behavior, which moreover facilitates an analytical understanding of the general PTD observed in the simulations of the *MG*. Furthermore, interesting properties arise which are only partially present in the *MG*, like the *quasi-periodic* behavior of the sequence of minority sides, which turns out to be *periodic* for the MG^{prior}.

Just like in the *MG*, we can also define the Full Strategy Minority Game with the prior choosing rule of strategies ($FSMG^{prior}$) as the maximal instance of the MG^{prior} which verifies that I is the complete set of potential agents of the MG^{prior}. In Appendix A we prove that the $FSMG^{prior}$ necessarily verifies the SPTD for even occurrences of the states. Finally, we prove in Appendix B the equivalence between the MG^{prior} and the MG^{prior}_{rand} (i.e., MG^{prior} with random updating rule by following [9]) in terms of σ^2/N.

1.1. Additional definitions

We include here some specific definitions, notations and results about the *MG* model. The number of states of the system is $\mathcal{H} = 2^m$, and we denote by $S_{\mathcal{H}}$ the complete set of states. The number of strategies is $\mathcal{L} = 2^{\mathcal{H}}$, and the complete set of strategies $S_{\mathcal{L}}$ is known as the *Full Strategy Space*. At each step of the game, each agent plays using her best strategy (i.e., the one which has predicted the minority side the greatest number of times). To this end, the strategies of every agent are ranked according to the number of rounds that each one has correctly predicted the minority side. If the strategies are tied