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The Tutte polynomial of an infinite family of outerplanar, small-world and self-similar graphs

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HIGHLIGHTS

- We find recursive formulas for the Tutte polynomial of a family of small-world and self-similar graphs.
- We obtain the number of recurrent configurations of these graphs.
- We obtain the number of minimal recurrent configurations of these graphs.
- We obtain the number of indegree sequences of strongly connected orientations of these graphs.

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ABSTRACT

In this paper we recursively describe the Tutte polynomial of an infinite family of outerplanar, small-world and self-similar graphs. In particular, we study the Abelian Sandpile Model on these graphs and obtain the generating function of the recurrent configurations. Further, we give some exact analytical expression for the Tutte polynomial at several special points

1. Introduction

The Tutte polynomial of a graph, due to W.T. Tutte [1], is a two-variable polynomial which plays an important role in computer science, engineering, optimization, statistical physics, and biology. In a strong sense it contains every graphical invariant that can be computed by deleting and contraction. The Tutte polynomial can be evaluated at particular points (x, y) to give numerical graphical invariants, including the number of spanning trees, the number of connected spanning subgraphs, the dimension of the bicycle space and many more. The Tutte polynomial also specializes to a variety of single-variable graphical polynomials, including the chromatic polynomial, the reliability polynomial and the flow polynomial [2,3]. Furthermore, the Tutte polynomial has been widely studied in the field of statistical physics where it appears as the partition function of the q-state Potts model $Z_G(q, v)$ [4–6]. In fact, if G is a graph on n vertices then

$$T_G(x, y) = (x-1)^{-k(G)}(y-1)^{-n(G)}Z_G((x-1)(y-1), (y-1))$$

and so the partition function of the q-state Potts model is simply the Tutte polynomial expressed in different variables.

The Abelian Sandpile Model (ASM) is a dynamical system introduced in statistical physics in order to study self-organized criticality [7,8]. This model has been studied intensively, both in the physics and the mathematics literature. This model

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shows that a simple dynamics can lead to the emergence of very complex structures and drives the system towards a stationary state which shares several properties of equilibrium systems at the critical point; e.g. power law decay of cluster sizes and of correlations of the height-variables. It is known that the recurrent configurations of the ASM on a graph G (equivalently *G-parking functions* [9]) are counted by $T_G(1, 1)$ which is exactly the number of spanning trees of graph G. The following refinement is also true: the coefficient of y^k in $T_G(1, y)$ is the number of recurrent configurations at level k [10].

In this paper, we follow a combinatorial approach and use the self-similar structure to recursively investigate the Tutte polynomial of a deterministic network model which was introduced by Francesc Comellas and Alicia Miralles in Ref. [11] and was further studied in Ref. [12]. This network is outerplanar, modular, small-world and has clustering zero. It is worth mentioning that the q-state Potts model on recursively defined graphs was studied first by D. Dhar [13] and some related works can be found in Refs.[14–17].

By analyzing all the spanning subgraphs of this family of graphs, we give the recursive formulas for the Tutte polynomial of M_n. In particular, as special cases of the general Tutte polynomial, we get:

- the generating function of the recurrent configurations;
- the number of recurrent configurations (=the number of spanning trees);
- the number of minimal recurrent configurations (=the number of acyclic orientations with a unique sink);
- the number of indegree sequences of strongly connected orientations;
- the number of strongly connected orientations;
- the number of acyclic orientations.

The paper is organized as follows. In Section 2, we recall some definitions and relevant results about a network model, the Tutte polynomial and the ASM. In Section 3, we study this model and give recursive formulas for the Tutte polynomial. In Section 4, we highlight the connections between the ASM and the Tutte polynomial on the model. In Section 5, we use these formulas to calculate the Tutte polynomial at several special points and give exact solutions for some enumerations on the graph M_n.

2. Preliminaries

The graph terminology used in this paper is standard and generally follows Diestel [18]. $G = (V(G), E(G))$ denotes a graph with vertex set $V(G)$ and edge set $E(G)$. The vertices a an b are the end-points of an edge $\{a, b\}$. An *orientation* of graph G is the digraph defined by the choice of a direction for every edge of $E(G)$. A *directed cycle* of a digraph is a set of edges forming a cycle of the graph such that they are all directed accordingly with a direction for the cycle. A digraph is *acyclic* if it has no directed cycle, and *strongly connected* if for every pair of vertices there is a directed cycle containing them. A *sink* for a digraph is a vertex with no outgoing edge. The *indegree sequence* of a orientation is a mapping defined on V associating with $v \in V$ the indegree of v.

2.1. The Tutte polynomial

There are several alternative definitions of the Tutte polynomial [2,3]. Here we present the definition in term of rank-nullity generating function.

A subgraph H of G is *spanning* if $V(H) = V(G)$. In particular, a *spanning tree* of G is a spanning subgraph of G which is a tree. Let H be a spanning subgraph of G and $k(H)$ be the number of connected components of H. We recall that the *rank* $r(H)$ and the *nullity* $n(H)$ of H are defined by

$$r(H) = |V(H)| - k(H) \quad \text{and} \quad n(H) = |E(H)| - r(H) = |E(H)| - |V(H)| + k(H).$$

Definition 2.1. Let $G = (V, E)$ be a graph. The Tutte polynomial $T_G(x, y)$ of G is defined by

$$T_G(x, y) = \sum_{H \subseteq G} (x-1)^{r(G)-r(H)} (y-1)^{n(H)},$$

where the sum runs over all the spanning subgraphs H of G.

2.2. The graph family $\{M_n\}$

This family of graphs $\{M_n\}$ was introduced by Francesc Comellas and Alicia Miralles in Ref. [11], and can be constructed by the following recursive process,

For $n = 0$, M_0 has two vertices and a edge connecting them.

For $n = 1$, M_1 is obtained from two copies of graph M_0 connected by two new edges which are not adjacent.

For $n \geq 2$, M_n is obtained from two copies of graph M_{n-1} by connecting them with two new edges. In each copy, there are two vertices which are chosen to connect with the other copy, such that the edge connecting the two chosen vertices is a new edge at step $n - 1$.