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# Deterministic self-similar models of complex networks based on very symmetric graphs

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## HIGHLIGHTS

- Generalized model of a self-similar deterministic network.
- Eight new infinite classes of complex networks with prescribed clustering coefficient.
- New deterministic self-similar, scale-free small world networks.

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## ABSTRACT

Using very symmetric graphs we generalize several deterministic self-similar models of complex networks and we calculate the main network parameters of our generalization. More specifically, we calculate the order, size and the degree distribution, and we give an upper bound for the diameter and a lower bound for the clustering coefficient. These results yield conditions under which the network is a self-similar and scale-free small world network. We remark that all these conditions are posed on a small base graph which is used in the construction. As a consequence, we can construct complex networks having prescribed properties. We demonstrate this fact on the clustering coefficient. We propose eight new infinite classes of complex networks. One of these new classes is so rich that it is parametrized by three independent parameters.



## 1. Introduction

In last years, many real-life networks from very different areas were studied, see e.g. Ref. [1], and it was observed that, typically, these networks have some common properties. They have small average degree, small distances between the vertices and big clustering. More precisely, if $n$ is the order (the number of vertices) of the network then:

(A1) The number of edges is in $O(n \ln n)$.
(A2) The diameter is in $O(\ln n)$.
(A3) For the clustering coefficient $C(G)$ we have $C(G) \geq c$ for some positive constant $c$.

These three properties define *small world networks* as proposed by Watts and Strogatz in Ref. [2]. Later, Barabási and Albert observed that many complex networks are scale-free; see Ref. [3]. More precisely:

(A4) The proportion of vertices of degree at least $k$ is approximately equal to $k^{1-\gamma}$, where $\gamma$ typically satisfies $2 < \gamma \leq 3$.

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Next property of complex-networks is self-similarity; see Ref. [4].

There appeared many random models of networks satisfying (A1)–(A4) and also several deterministic ones. The deterministic models, whose advantage is that their properties can often be computed analytically, have usually some common features. We define here a general construction, such that the networks introduced in Refs. [5–9], and with a slight modification also those of [10–13], are special cases of our construction. However, there are deterministic constructions, such as the hierarchic models of Refs. [14,15] and the model based on edge valuations [16], which we do not generalize.

After introducing our model, we calculate the number of vertices and edges of the network, we find the degree distribution and we calculate $\gamma$ from (A4). We also find a good upper bound for the diameter and a lower bound for the clustering coefficient, so that (A2) and (A3) can be checked easily. Then we demonstrate our results on three previously invented models. Our choice of these models was such that they are as different as possible. As expected, all our results agree for these networks.

Finally, based on our general construction, we introduce eight new deterministic models of complex networks. All our constructions yield infinite classes of networks, and one of them, namely Construction 6, can be parametrized by three independent parameters. Non self-repetitive version of another one, namely Construction 5, generalizes both the Farey network and the Apollonian network constructions; see Refs. [6,17]. In the process of modelling these constructions we focused our attention to the clustering coefficient. Three of these new constructions have clustering coefficient close to 1 and three have this coefficient close to $\frac{1}{2}$. The first three are based on the complete graph $K_k$ and the second three are based on the complete tripartite graph $K_{k,k,k}$. The last two constructions have clustering coefficient 0. One is based on the complete bipartite graph $K_{k,k}$ and one on the graph of a prism. These last two constructions do not satisfy (A3), but we include them here as some of the previous models of self-similar networks also have clustering coefficient 0 (to demonstrate this fact see Constructions 1 and 2 below). We point out that in all these new models all the parameters mentioned above are obtained by simple substitution of relevant constants to the derived formulae.

## 2. Three previously invented models

We recall here three constructions of self-similar networks. The first construction appeared in Zhang and Comellas [18], the second in Comellas, Zhang and Chen [19] and the third in Comellas, Fertin and Raspaud [20]. However, we define these constructions in slightly different words than in Refs. [18–20]. The reason for this is that we like to point out their common features, which are generalized in the next section. For illustration of these constructions see Fig. 1.

**Construction 1.** Let $H$ be a graph with one special edge $\mathcal{U}$, end-vertices of which are connected by $j$ internally-vertex-disjoint paths of length 3. Thus, if $j = 1$ then $H$ is a square, while if $j > 1$ then $H$ consists of $j$ squares sharing the edge $\mathcal{U}$. Denote by $\mathcal{T}$ the set of edges of $H$ that contain exactly one vertex of $\mathcal{U}$. Then $\mathcal{T}$ has $2j$ edges and we call them active edges. The construction is following:

- If $t = 0$, $G_j(0)$ has two vertices connected by an active edge.
- If $t > 0$, $G_j(t)$ is obtained from $G_j(t-1)$ by identifying every active edge of $G_j(t-1)$ with the edge $\mathcal{U}$ of a copy of $H$. Hence, if $G_j(t-1)$ contains $q(t-1)$ active edges, we glue to $G_j(t)$ exactly $q(t-1)$ new copies of $H$. In $G_j(t)$, the active edges are exactly the edges of $\mathcal{T}$'s in just attached $q(t-1)$ copies of $H$.

**Construction 2.** Let $H$ be the graph of a cube, that is, $H$ has 8 vertices all of which have degree 3. Denote by $\mathcal{U}$ one square of $H$. Let $\mathcal{T}$ be the set of squares of $H$ which share exactly one edge with $\mathcal{U}$. Then $\mathcal{T}$ has four squares and we call them active squares. The construction is following:

- If $t = 0$, $G(0)$ consists of one active square.
- If $t > 0$, $G(t)$ is obtained from $G(t-1)$ by identifying every active square of $G(t-1)$ with the square $\mathcal{U}$ of a copy of $H$, so that the edges of $\mathcal{U}$ are identified with the edges of the active square of $G(t-1)$. Hence, if $G(t-1)$ has $q(t-1)$ active squares, we glue to $G(t-1)$ exactly $q(t-1)$ copies of $H$. In $G(t)$ the active squares are exactly the squares of $\mathcal{T}$'s in just attached $q(t-1)$ copies of $H$.

**Construction 3.** Let $H$ be the complete graph on $j+1$ vertices, $K_{j+1}$, and let $\mathcal{U}$ be one of its induced subgraphs on $j$ vertices. Then $\mathcal{U}$ is a complete graph on $j$ vertices $K_j$. Denote by $\mathcal{T}$ the set of all induced $j$-vertex subgraphs of $H$, including $\mathcal{U}$ itself. The complete graphs of $\mathcal{T}$ are active. The construction is following:

- If $t = 0$, $G_j(0)$ consists of an active complete graph on $j$ vertices.
- If $t > 0$, $G_j(t)$ is obtained from $G_j(t-1)$ by identifying every active $K_j$ with $\mathcal{U}$ of a copy of $H$. Hence, if $G_j(t-1)$ has $q(t-1)$ active $K_j$'s, we glue to $G_j(t-1)$ exactly $q(t-1)$ copies of $H$. In $G_j(t)$, active copies of $K_j$ are exactly the graphs of $\mathcal{T}$'s in just attached copies of $H$.

In our notation, $G(0)$ is isomorphic to $\mathcal{U}$ and $G(1)$ is isomorphic to $H$ in Constructions 1–3. In Construction 3, if some copy of $K_j$ is active once, then it is active forever. Therefore we call Construction 3 *self-repetitive*, while Construction 1 and Construction 2 are *non self-repetitive*. In [20], Construction 3 is defined in a slightly different way. It says that in $t$-th iteration we find all the copies of $K_j$ in $G_j(t-1)$ and to all vertices of every such $K_j$ we connect one new vertex. But it is easy to see that our definition is equivalent with this one.