Physica A 392 (2013) 3045–3050

Contents lists available at SciVerse ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

A generalization of the q-exponential discounting function[☆]

Salvador Cruz Rambaud [1], María José Muñoz Torrecillas [*]

Departamento de Economía y Empresa, University of Almería, Spain

HIGHLIGHTS

- We generalize the q-exponential discounting function introduced by Cajueiro.
- We extend it to any discounting function, including dynamic discounting functions.
- We extend the domain of the q parameter to the joint interval $(\infty, 1) \cup (1, +\infty)$.
- Subadditive or superadditive discounting functions are generated from different q.
- Interdisciplinary approach: q-exponential is inspired in Tsallis thermodynamics.

ARTICLE INFO

Article history:
Received 11 December 2012
Received in revised form 15 February 2013
Available online 28 March 2013

Keywords:
Discounting
Inconsistency
Hyperbolic
Exponential
Subadditive
Econophysics

ABSTRACT

The aim of this paper is to generalize the q-exponential discounting function introduced by Cajueiro (2006) [1] using the hyperbolic function as a base. The presented generalization has two aspects. First, we consider any discounting function $F(t)$, and not just hyperbolic discounting. Second, the value of the parameter q is extended to the joint interval $(-\infty, 1) \cup (1, +\infty)$. In this way, we have found a family of discounting functions whose elements are subadditive or superadditive according to the value of q.

1. Introduction

Cajueiro [1] introduces the *q-exponential discounting function* as the inverse (with respect to the multiplication of functions) of the q-exponential function[2]:

$$\beta_q(t) = \frac{1}{[1 + (1-q)\alpha t]^{1/(1-q)}}, \tag{1}$$

with $q \in [0, 1)$, this parameter represents a measure of consistency in intertemporal decisions quantified by the discounting function. In effect,

☆ This paper has been partially supported by the project "Valoración de proyectos gubernamentales a largo plazo: obtención de la tasa social de descuento", reference: P09-SEJ-05404, Proyectos de Excelencia de la Junta de Andalucía and Fondos FEDER. The authors are very grateful for the comments and suggestions of two anonymous referees.
* Corresponding author. Tel.: +34 950 015 817; fax: +34 950 015 178.
E-mail addresses: scruz@ual.es (S. Cruz Rambaud), mjmtorre@ual.es (M.J. Muñoz Torrecillas).
1 Tel.: +34 950 015 184; fax: +34 950 015 178.
2 This function has been inspired in Tsallis's [2] nonextensive thermodynamics and has recently been introduced in econophysics and neuroeconomics studies.

0378-4371/$ – see front matter
http://dx.doi.org/10.1016/j.physa.2013.03.009

- When $q \to 1$ then $\beta_q(t)$ tends to the classical *exponential discounting* and the intertemporal choices are consistent.
- When $q = 0$ then $\beta_q(t)$ is *hyperbolic discounting*. In this case and in all the cases where q belongs to the interval $[0, 1)$ the intertemporal choices are inconsistent.

The q-parameter proposed by Cajueiro [1] is then contained in the interval $[0, 1)$. Nevertheless, Takahashi [3] proposes an extension of the domain of q in order to allow for the q-parameter to be negative. The estimated q-values in his experiments were negative "indicating that participants' typical intertemporal choice was 'more hyperbolic', i.e., more inconsistent discounting than hyperbolic discounting (corresponding to $q = 0$ in the q-discount model)". Estimated q-values were -2.63 for oneself and -8.89 for someone else, indicating a more inconsistent intertemporal choice behaviour when the outcomes of intertemporal choice were only relevant to someone else[3] than when relevant to oneself.

Experimental values of q were also smaller than zero in Ref. [4], i.e., the median value of q for the group data was -3.87. In this paper Takahashi finds that q-exponential discounting better fits group data whilst quasi-hyperbolic discounting better fits individual data.

In Ref. [5] the estimated values of q were also out of the range $[0, 1)$ that Cajueiro defined for the q-exponential discounting function. In this paper, some anomalies in intertemporal choice are examined, namely hyperbolic discounting, which results in time preference reversal, and the sign effect,[4] by utilizing a q-exponential temporal discounting model. They obtained a q-parameter of -6.54 in their *time discount model with physical time*, which they interpret as the time discount functional form being closer to the hyperbolic model than the exponential model. This value of q was even smaller (-61.8) for the case of *time discounting with valuation of outcomes*. In this case, they measured one's subjective affect (subjective valuation) towards positive and negative outcomes, i.e., "happiness" when obtaining monetary gains and "aversion" to monetary losses. The q parameter for the case of *time discounting with psychological time*[5] was equal to 3.72 and this is interpreted as the functional form being closer to the exponential rather than the hyperbolic model. This value of q was even higher (27.1) for the case of *time discounting models with psychological time and valuation of outcomes*.

In this paper, we are going to generalize the q-exponential discounting function in order to include, in the same mathematical expression, consistency and inconsistency, when the latter is due to either subadditivity or superadditivity. In the context of subadditivity or superadditivity, dynamic discounting functions are essential since the criterion of intertemporal choice is variable depending on the benchmark or point of reference [7].

Following Green and Myerson's [8] definition of inconsistency, consider a smaller-sooner reward and a larger-later reward. When the delays are relatively early, the subjective value of the smaller reward is higher than the value of the bigger reward; nevertheless, when both delays are relatively late, the bigger reward (the more distant in time) has the higher subjective value, giving rise to inconsistency. In particular, hyperbolic discounting implies higher discount rates for short periods of time than for long periods. So, intertemporal choices made using hyperbolic discounting will be inconsistent. On the other hand, exponential discounting is a particular case of consistent intertemporal choice, since it predicts that there will not be preference reversal. So, agents will prefer the smaller-sooner reward, independently of the moment of decision making.

A source of inconsistency can be given by the subadditivity of the discounting function used. Subadditive discounting[6] means that the discount is higher when the interval is divided into subintervals. Subadditive discounting implies smaller values of the discounting function for more subdivided intervals. For example, the discounting function for one year will be greater than the product of the corresponding discounting function values for each month. On the other hand, superadditive discounting means that the discount is smaller when the interval is divided into subintervals and therefore the discounting function will be greater for more subdivided intervals.

The organization of this paper is as follows. After this introduction, Section 2 introduces the general deformation $F_q(t) = [F((1-q)t)]^{1/(1-q)}$, where $F(t)$ is any discounting function and moreover $q \in (-\infty, 1) \cup (1, +\infty)$. Observe that this is the deformation proposed by [1] where hyperbolic discounting has been substituted by $F(t)$ and where the interval has been extended to $(-\infty, 1) \cup (1, +\infty)$. Here we show that non-additivity is an invariance for this deformation. In Section 3 all the general results obtained in Section 2 are applied to the case in which $F(t)$ is hyperbolic discounting. Section 4 gives some indications when deforming a dynamic discounting function $F(d, t)$, where d is the time as a date (delay) and t is the time as an interval. Finally, Section 5 summarizes and concludes.

2. A deformed discounting function based on the *q*-exponential

Definition 1. A *stationary discounting function* $F(t)$ is a continuous real function defined within an interval $[0, t_0)$ (t_0 can even be $+\infty$) satisfying the following conditions:

---

3 A clear example of this type of choice performed by someone else is governmental economic policy-making.

4 The sign effect or gain–loss asymmetry implies lower discount rates for losses than for gains. This and other anomalies are reviewed in Ref. [6].

5 They fitted three psychophysical models (Steven's power, Weber–Fechner logarithm and linear model) to psychological time in waiting for delayed gain and loss. To examine psychological time empirically, participants were asked to indicate the length of psychological time ("very short" to "very long") until they receive (or pay) money.

6 Refs. [9–12].