Physica A 392 (2013) 1648–1658

Contents lists available at SciVerse ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Default risk modeling with position-dependent killing

Yuri A. Katz

S&P Capital IQ, 55 Water Str., New York, NY 10041, United States

ARTICLE INFO

Article history:
Received 26 April 2012
Received in revised form 4 October 2012
Available online 21 December 2012

Keywords:
Credit risk
Default probability
Position-dependent dissipation

ABSTRACT

Diffusion in a linear potential in the presence of position-dependent killing is used to mimic a default process. Different assumptions regarding transport coefficients, initial conditions, and elasticity of the killing measure lead to diverse models of bankruptcy. One "stylized fact" is fundamental for our consideration: empirically default is a rather rare event, especially in the investment grade categories of credit ratings. Hence, the action of killing may be considered as a small parameter. In a number of special cases we derive closed-form expressions for the entire term structure of the cumulative probability of default, its hazard rate, and intensity. Comparison with historical data on aggregate global corporate defaults confirms the validity of the perturbation method for estimations of long-term probability of default for companies with high credit quality. On a single company level, we implement the derived formulas to estimate the one-year likelihood of default of Enron on a daily basis from August 2000 to August 2001, three months before its default, and compare the obtained results with forecasts of traditional structural models.

1. Introduction

Events on stock and credit markets are interrelated, complex, and fundamentally indeterminate. Nevertheless, similarly to natural sciences and engineering, an adequate representation of phenomena in financial economy may be obtained by using certain phenomenological constructs augmented by measurements of relevant "macroscopic" variables. For instance, in survival analysis and reliability theory the likelihood of destruction is characterized by the relevant time-dependent hazard rate function [1,2]. It determines the risk of failure as a function of time, conditional on not having happened previously. In essence, it describes "how the past affects the future" and, hence, is the main characteristic of a generally non-Markovian evolution of a system. Similarly, in finance the hazard rate function characterizes the risk of default, i.e., failure to pay financial obligations, and is the key attribute of the phenomenological "reduced-form" framework [3,4]. The description of default via the hazard rate leads to closed-form pricing formulas that can be calibrated to fit observable prices of credit risk sensitive securities. This procedure enables measurements of the market-implied ("risk-neutral") probability of default (PD), which can be used to estimate prices of illiquid securities bearing similar credit risk [5].

As any phenomenological approach, the reduced-form framework is applicable without any explanation of the "microscopic" causes of default. Similarly to laws of thermodynamics, which were put into practice well before statistical physics, it may be used without any microscopic justification. Yet, even an approximate understanding of a default mechanism and its relationship to empirical regularities are very important. Only a self-consistent theory allowing for *derivation* of the hazard rate function from some microscopic modeling assumptions may validate a priori specifications that have been employed, e.g., in a number of "jump-to-default" equity derivative models [6,7]. Verification of these conjectures is impossible within the reduced-form approach. Similar to traditional structural models, such a theory should be based on clear economic micro-foundations, provide insight and intuition, and allow for differentiation between alternative mechanisms.

E-mail address: yuri_katz@sandp.com.

0378-4371/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.physa.2012.11.059

It is remarkable in this context that almost a century ago Smoluchowski had derived the closed-form expression for a hazard (recombination) rate function of an irreversible chemical reaction assisted by diffusion of reagents [8,9]. He established the fundamental connection between the conditional probability density function (pdf) describing a relative position of microscopic particles and the phenomenological hazard rate function. The problem was solved within the first-passage approximation, which implies that a diffusive path of reagents stops at any encounter. Similarly, in the Black–Cox extension of the original Merton model [10] default happens at the first passage, i.e., irreversibly, instantly, and at any contact, whenever the diffusive path of firm's assets value hits the *absorbing* default barrier. In this model the risk of default is completely determined by the probability of issuer's assets to attain an extreme value – equal or below a critical level of its total liabilities – for the first time. This assumption has been customarily employed in traditional structural models of default, which include non-zero coupon bonds [11], stochastic interest rates [12], and endogenously defined default boundaries [13]. We should distinguish, however, between a contact with the default boundary and actual destruction. There is always some uncertainty related to the issuer's ability to avoid default even if liabilities momentarily exceeding its assets. In other words, not every encounter with the default barrier leads to bankruptcy. In the recent paper Katz and Shokhirev [14], by using an analogy with the Collins–Kimball recombination model [15], introduced the local rate of default at the boundary. The latter may depend on industry sector, country, region, macro-economic environment, business cycle, etc. The extended Black–Cox (EBC) model of default with the *radiation* boundary condition [14] does not fully prescribe the future of any particular encounter. Yet, it has a fundamental drawback related to the assumed contact character of default, which obviously leads to zero hazard rate if the boundary is inaccessible. This limitation of the contact approximation can be relaxed only in the theory incorporating a position-dependent dissipative term into transport equations [16]. Nowadays, different flavors of this theory are used to describe a variety of processes in physics, chemistry, and molecular biology [17,18].

It is the purpose of this paper to apply the basic ideas and formalism developed in these studies to valuation of a default risk. Our approach draws on insights gained from the works of Duffie and Lando [19], Jarrow and Protter [20] and other authors [21–28] that are based on the postulate of incomplete information regarding key financial parameters of a firm. Systematic misreporting of key financial parameters in the pre-default time has been clearly demonstrated recently by Podobnik et al. [29]. These authors have shown that an issuer may default in a wide range of distances to the default boundary (leverage ratios). Therefore, it is important to analyze the role of "remote killing" in valuation of a default risk. Here we assume that default is characterized by the position-dependent (*elastic*) killing term introduced into the Fokker–Planck equation (FPE). The latter, in accordance to standard structural models, describes the geometric Brownian motion of a pre-default firm's assets value. Introduction of an elastic killing measure brings the concept of incomplete information and indeterminacy of a default event into the Black–Scholes–Merton framework. This approach naturally generalizes the contact approximation and, hence, allows for checking the limits of its validity. Herein we re-derive the main results of the contact model of default [14] under the assumption that the spatial distribution of the killing measure is reduced to the Dirac delta-function. Additionally, we obtain the approximate analytical expressions for the intensity of default and the term structure of the cumulative PD under the assumption that elasticity of the killing measure is determined by the Gaussian density function. One "stylized fact" is fundamental for our quantitative consideration: empirically bankruptcy is a rather rare (extreme) event, especially in the investment grade categories of credit ratings. Hence, the *action* of killing may be considered as a small parameter of the problem. To apply the perturbation method we transform the FPE with the position-dependent dissipative term into the integral equation. The latter is formally equivalent to the Feynman integral equation for quantum propagators [30] with a scattering potential replaced by an elastic killing term. The new formulation of the problem facilitates the robust perturbation expansion relating the Green functions of the FPEs with and without dissipation, which may be useful for valuation of different types of risk. Comparison between obtained formulas and historical data on aggregate global corporate defaults [31] confirms the validity of the perturbation method for companies with high credit quality. On a single company level, we implement the EBC model to estimate the one-year likelihood of default of Enron on a daily basis from August 2000 to August 2001, three months before its default, and compare the obtained results with forecasts of traditional structural models.

2. The framework

Consider the single stochastic state variable $x = -\ln R = \ln(V/L)$, which represents the company's position in "space". Here R is the firm's leverage defined as the ratio between firm's total liabilities L and its aggregate asset's value V. Following the traditional structural modeling approach, let us assume that under the physical (actual) probability measure the stochastic dynamics of x satisfy the Ito differential equation with generally position-dependent transport coefficients [4,5]

$$\mathrm{d}x = a(x)\mathrm{d}t + \sigma(x)\mathrm{d}W. \tag{2.1}$$

Here $a(x) = \mu(x) - \sigma^2(x)/2$ is the effective drift, $\mu(x)$ is the rate of change in a firm's leverage ratio, $\sigma(x)$ is the volatility of a company's assets value, and W represents the standard Wiener process. Consequently, the conditional pdf $p(x, t|X_0, 0) \equiv p(x, t|x_0)$ satisfies the following one-dimensional FPE

$$\partial_t p(x, t|x_0) = -\nabla_x j(x, t|x_0), \tag{2.2}$$