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Kinetics of node splitting in evolving complex networks

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ABSTRACT

We introduce a collection of complex networks generated by a combination of preferential attachment and a previously unexamined process of "splitting" nodes of degree k into k nodes of degree 1. Four networks are considered, each evolves at each time step by either preferential attachment, with probability p, or splitting with probability $1-p$. Two methods of attachment are considered; first, attachment of an edge between a newly created node and an existing node in the network, and secondly by attachment of an edge between two existing nodes. Splitting is also considered in two separate ways; first by selecting each node with equal probability and secondly, selecting the node with probability proportional to its degree. Exact solutions for the degree distributions are found and scale-free structure is exhibited in those networks where the candidates for splitting are chosen with uniform probability, those that are chosen preferentially are distributed with a power law with exponential cut-off.

1. Introduction

A considerable literature has been established studying complex networks and their application to various natural and social phenomena. Much of this research concentrates on simple stochastic processes which, when repeated a large number of times, generate complex networks with a variety of topological structures and characteristics. At the simplest level these processes are limited to the addition and removal of nodes and edges. Interactions amongst agents in many naturally occurring systems have successfully been described in such a way, consequently a broad range of possible network evolutions are now understood [1].

In *preferential attachment* a new node is created at each time-step and linked to m existing nodes in the network, by design the likelihood of linking to a node of degree k is proportional to k. After many iterations the proportion of nodes which have degree k has been shown to have power law behaviour: $P(k) \sim 2m^2k^{-3}$ where $P(k)$ is the proportion of nodes having degree k [2]. Many real-world systems have been explained with this model. Most successfully perhaps is the network of citations in scientific publications, papers here are represented by nodes and links are formed between each paper and those papers it cites, empirical data confirms the power law exponent with remarkable accuracy [3]. An extension of this model incorporates the addition of links between existing nodes [4], originally introduced to describe the social network of scientific collaborations though similar variations have also been used to model the interactions of words in human language [5]. The degree distribution still follows a power law although it is now composed of two regimes divided by a critical point where the exponent changes.

In the broader field of statistical mechanics, a substantial body of research concerns the coalescence and fragmentation of clusters of particles. Applications in this field span a variety of subjects including astrophysics [6], polymerisation [7] and aerosols [8]. Despite the diversity of applications the basic model remains the same; two clusters containing either a number, in the discrete case, or mass in the continuous, of identical particles of sizes x and y *coalesce* at a rate $K(x, y)$ into a cluster

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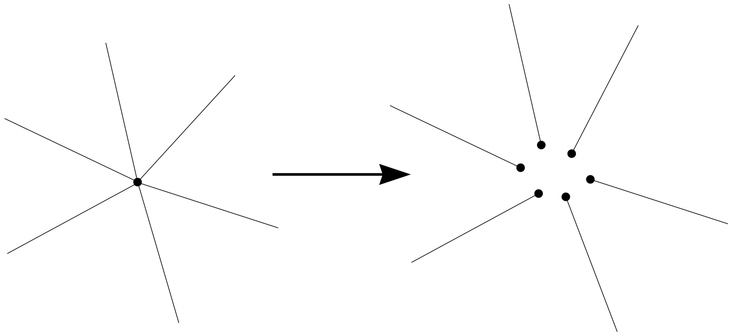

Fig. 1. The effect of "splitting" a node of degree 6 in a network.

of size $x + y$. In the discrete setting, the number of clusters of size x at time t denoted by $n(x, t)$ obeys the *Smolochowski coagulation equation*,

$$\frac{\partial}{\partial t}n(x,t) = \frac{1}{2}\sum_{y=1}^{x-1} K(y, x-y)n(y,t)n(x-y,t) - n(x,t)\sum_{y=1}^{\infty} K(x,y)n(y,t) \quad (1)$$

where the first term on the right hand side accounts for the creation of a cluster of size x from the coalescence of two smaller clusters and the second term accounts for the loss of a cluster of size x when it coalesces with another. Exact general solutions have not been found, however in the special cases where $K(x, y) = 1$ for example, representing two clusters coalescing at each time step, and $K(x, y) = xy$, where clusters coalesce at a rate proportional to their size, exact solutions do exist [9]. Conversely, equivalent equations for fragmentation are constructed in a similar way. If coalescence and fragmentation are simultaneously present in a model then complete fragmentation or complete coalescence into one supercluster can be avoided; in this case the distribution of cluster sizes at large t is independent of t. A model of this type has been used to describe the herding behaviour of traders in financial markets [10], here traders are the particles of the system and clusters represent groups of traders sharing information and therefore trading in the same way. The clusters of this model coalesce over time and at random times will rapidly fragment into unclustered individuals. It was shown that the size of the clusters at large t follows a power-law distribution with exponential cut-off, it has also been proposed as a possible reason why variations in share price do not follow a Gaussian distribution [11]. The networks presented in this paper extend the lexicon of complex networks by translating the previous model into a network environment. By considering link formation between nodes to be equivalent to coalescence, and by introducing a new process that we shall refer to as "splitting" to parallel the fragmentation process described above, we reproduce the cluster size distribution as a network degree distribution.

We define splitting as the replacement of a single node of degree k with k nodes of degree one (see Fig. 1) and examine the topologies of networks created through this splitting process alongside other growth processes. The evolution of the networks studied here is also driven by the preferential attachment mechanisms outlined in Ref. [12], first in Section 2, where new nodes are linked to existing nodes in the network chosen with probability proportional to their degree, and secondly in Section 3 edges are attached between pairs of existing nodes, again with probability proportional to their degree.

We use $C_k(t)$ to denote the number of nodes of degree k at time t and introduce the following two quantities

$$N(t) = \sum_{k=1}^{\infty} C_k(t) \quad (2)$$

and

$$M(t) = \sum_{k=1}^{\infty} kC_k(t) \quad (3)$$

where $N(t)$ is the total number of nodes and $M(t)$ is the total degree of the network at time t.

2. Node attachment model

At each time step the network may develop in one of the two following ways:

(a) With probability p a node is attached by an edge to an existing node, the probability that the end of the edge attaches to a node of degree k is proportional to k.
(b) With probability $1 - p$ a node of degree k is randomly selected and split into k nodes of degree 1.