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Geometry of deformed exponential families: Invariant, dually-flat and conformal geometries

Shun-ichi Amari [a], Atsumi Ohara [b,*], Hiroshi Matsuzoe [c]

[a] Riken Brain Science Institute, Wako-shi Hirosawa 2-1, Saitama 351-0198, Japan

[b] Department of Electrical and Electronics Engineering, University of Fukui, Bunkyo 3-9-1, Fukui 910-8507, Japan

[c] Department of Computer Science and Engineering, Graduate School of Engineering, Nagoya Institute of Technology, Gokiso-cho, Showa-ku, Nagoya 466-8555, Japan

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ABSTRACT

An information-geometrical foundation is established for the deformed exponential families of probability distributions. Two different types of geometrical structures, an invariant geometry and a flat geometry, are given to a manifold of a deformed exponential family. The two different geometries provide respective quantities such as deformed free energies, entropies and divergences. The class belonging to both the invariant and flat geometries at the same time consists of exponential and mixture families. The q-families are characterized from the viewpoint of the invariant and flat geometries. The q-exponential family is a unique class that has the invariant and flat geometries in the extended class of positive measures. Furthermore, it is the only class of which the Riemannian metric is conformally connected with the invariant Fisher metric.

1. Introduction

Since the introduction of q-entropy by Tsallis [1] (see also an extensive monograph [2]), much attention has been paid to non-extensive statistical mechanics. It is related to various 'non-standard phenomena' subject to the power law not only in statistical physics but in economics and disaster statistics. Here, families of probability distributions of the q-exponential family and more general deformed exponential families play a major role. In the present paper, a geometrical foundation is given to these families of distributions from the point of view of information geometry [3].

The deformed exponential family was introduced and studied extensively by Naudts [4,5] (see also a monograph [6]). Kaniadakis et al. [7] studied the κ-exponential family which belongs to the deformed exponential family. Its mathematical structure was studied by Pistone [8] and Vigelis and Cavalcante [9]. See other examples with interesting discussions [10,11]. In statistics, a similar notion of a generalized exponential family [12] or the U-model [13,14] is discussed on the bases of respective motives.

Many useful concepts such as generalized entropy, divergence and escort probability distribution have been proposed. However, their relationships have not necessarily been well understood theoretically and are waiting for further geometrical and statistical elucidation. It is also useful to characterize the q-families in the class of general deformed exponential families.

In the present study, information geometry [3] is used to give a foundation to the deformed exponential families. Two types of geometry can be introduced in the manifold of a deformed exponential family: One is the *invariant geometry*, where the Fisher information is the unique Riemannian metric (Chentsov [15]; also see Ref. [3]) together with a dual pair of invariant affine connections (α-connections). The other is the *dually flat geometry* [3] (also see Ref. [16]), which is not necessarily

* Corresponding author.
E-mail addresses: amari@brain.riken.jp (S.-i. Amari), ohara@fuee.u-fukui.ac.jp (A. Ohara), matsuzoe@nitech.ac.jp (H. Matsuzoe).

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invariant but accompanies the Legendre structure. The escort probability distribution belongs to the latter geometry. The two geometries give different free-energies, entropies and divergences in general.

The exponential and mixture families are characterized by the property that they sit at the intersection of the classes of the invariant and flat geometries. The q-exponential family is then characterized from two viewpoints: One is invariance and flatness in the class of positive measures and the other is conformal geometry [17,18]. It is shown that the q-family is a unique class of flat geometry that is connected conformally to the invariant geometry.

2. Deformed exponential family

We follow Naudts [6,19] for the formulation of the deformed exponential family. Given a positive increasing function $\chi(s)$ on $(0, \infty) \in \boldsymbol{R}$, a deformed logarithm, called the χ-logarithm, is defined by

$$\ln_\chi(s) = \int_1^s \frac{1}{\chi(t)} \mathrm{d}t. \tag{1}$$

This is a concave monotonically increasing function. When χ is a power function,

$$\chi(s) = s^q, \quad q > 0, \tag{2}$$

(1) gives the q-logarithm

$$\ln_q(s) = \frac{1}{1-q}\left(s^{1-q} - 1\right). \tag{3}$$

The ordinary logarithm is obtained by taking the limit of $q = 1$.

The inverse of the χ-logarithm is the χ-exponential, given by

$$\exp_\chi(t) = 1 + \int_0^t \lambda(s)\mathrm{d}s, \tag{4}$$

where $\lambda(s)$ is defined by the relation

$$\lambda\left\{\ln_\chi(s)\right\} = \chi(s). \tag{5}$$

The q-exponential is given by

$$\exp_q(t) = \{1 + (1-q)t\}^{\frac{1}{1-q}}, \tag{6}$$

where the limit $q = 1$ gives the ordinary exponential.

A family $S = \{p(\boldsymbol{x}, \boldsymbol{\theta})\}$ parameterized by $\boldsymbol{\theta} = \left(\theta^1, \dots, \theta^n\right)$ of probability distributions of a vector random variable $\boldsymbol{x} = (x_1, \dots, x_n)$ is called a χ-exponential family, when its density function is given by

$$p(\boldsymbol{x}, \boldsymbol{\theta}) = \exp_\chi\left\{\sum_i \theta^i x_i - \psi(\boldsymbol{\theta})\right\} \tag{7}$$

with respect to a dominating measure $\mu(\boldsymbol{x})$. Here, $\psi(\boldsymbol{\theta})$ is determined from the normalization condition

$$\int p(\boldsymbol{x}, \boldsymbol{\theta}) d\mu(\boldsymbol{x}) = 1 \tag{8}$$

and is called the χ-free energy. Family S is regarded as an n-dimensional manifold, where $\boldsymbol{\theta}$ plays the role of a coordinate system. We call $\boldsymbol{\theta}$ the χ-coordinate system of the χ-exponential family. Any linear subspace in the $\boldsymbol{\theta}$-coordinates is also a χ-exponential family. The q-exponential family is χ-exponential family given by

$$p(\boldsymbol{x}, \boldsymbol{\theta}) = \exp_q\left\{\sum \theta^i x_i - \psi(\boldsymbol{\theta})\right\}. \tag{9}$$

Let us consider discrete random variable x, taking values on $X = \{0, 1, \dots, n\}$. Let $S_n = \{p(x)\}$ be the family consisting of all such probability distributions. It is called the probability n-simplex. By introducing a new vector random variable $\boldsymbol{x} = \{\delta_i(x)\}$,

$$\delta_i(x) = \begin{cases} 1, & x = i \\ 0, & \text{otherwise}, \end{cases} \tag{10}$$

we have

$$p(x) = \sum_{i=0}^n p_i \delta_i(x), \tag{11}$$

where $p_i = \mathrm{Prob}\,\{x = i\}$ with constraint $\sum p_i = 1$. Hence, S_n is an n-dimensional manifold, where $\boldsymbol{p} = (p_1, \dots, p_n)$ plays the role of a coordinate system.