Available online at www.sciencedirect.com
SCIENCE DIRECT
Discrete Optimization 2 (2005) 113–122
DISCRETE OPTIMIZATION
www.elsevier.com/locate/disopt

# A semidefinite optimization approach for the single-row layout problem with unequal dimensions

Miguel F. Anjos[a,*,1], Andrew Kennings[b,2], Anthony Vannelli[b,3]

[a] *Operational Research Group, School of Mathematics, University of Southampton, Highfield, Southampton SO17 1BJ, UK*
[b] *Department of Electrical and Computer Engineering, University of Waterloo, Waterloo, Ont., Canada N2L 3G1*

Received 2 April 2004; received in revised form 29 March 2005; accepted 31 March 2005

**Abstract**

The facility layout problem is concerned with the arrangement of a given number of rectangular facilities so as to minimize the total cost associated with the (known or projected) interactions between them. We consider the one-dimensional space-allocation problem (ODSAP), also known as the single-row facility layout problem, which consists in finding an optimal linear placement of facilities with varying dimensions on a straight line. We construct a semidefinite programming (SDP) relaxation providing a lower bound on the optimal value of the ODSAP. To the best of our knowledge, this is the first non-trivial global lower bound for the ODSAP in the published literature. This SDP approach implicitly takes into account the natural symmetry of the problem and, unlike other algorithms in the literature, does not require the use of any explicit symmetry-breaking constraints. Furthermore, the structure of the SDP relaxation suggests a simple heuristic procedure which extracts a feasible solution to the ODSAP from the optimal matrix solution to the SDP relaxation. Computational results show that this heuristic yields a solution which is consistently within a few percentage points of the global optimal solution.


*Keywords:* Facilities planning and design; Space allocation; Combinatorial optimization; Semidefinite optimization; Global optimization

## 1. Introduction

The facility layout problem is concerned with the arrangement of a given number of rectangular facilities so as to minimize the total cost associated with the (known or projected) interactions between them. Versions of the facility layout problem occur in many environments, such as hospital layout and service center layout. This is a hard problem in general; most versions of this problem in the research literature are known to be NP-hard. A thorough survey of the facility layout problem is given in [24], where the research papers on facility layout are divided into three broad areas. The first is concerned with algorithms for tackling

* Corresponding author. Present address: Department of Management Sciences, University of Waterloo, Waterloo, Ont., Canada N2L 3G1.
*E-mail addresses:* anjos@stanfordalumni.org (M.F. Anjos), akenning@cheetah.vlsi.uwaterloo.ca (A. Kennings), vannelli@cheetah.vlsi.uwaterloo.ca (A. Vannelli).
1 Research of M.F. Anjos was partially supported by Grant NAL/00636/G from the Nuffield Foundation and Grant A2002/3 from the University of Southampton.
2 Research of A. Kennings was partially supported by the Natural Sciences and Engineering Research Council (NSERC) of Canada Discovery Grant 203763-03.
3 Research of A. Vannelli was partially supported by the Natural Sciences and Engineering Research Council (NSERC) of Canada Operating Grant 15296 and a Bell University Laboratories Research Grant.

1572-5286/$ - see front matter 
doi:10.1016/j.disopt.2005.03.001

the general layout problem as defined above. The second area is concerned with extensions of the problem in order to account for additional issues which arise in applications, such as designing dynamic layouts by taking time dependency issues into account; designing layouts under uncertainty conditions; and achieving layouts which optimize two or more objectives simultaneously. The third area is concerned with specially structured instances of the problem. One such special case that has been extensively studied occurs when all the facilities have equal dimensions and the possible locations for the facilities are given a priori; this is the quadratic assignment problem (QAP) formulated by Koopmans and Beckman [19]. The QAP assigns every facility to one location and at most one facility to each location, and the cost of placing a facility at a particular location is dependent on the location of the interacting departments. Since the possible locations are fixed, the problem reduces to optimizing a quadratic objective over all possible assignments of facilities to locations. The QAP is NP-hard, and is in general a hard problem to solve. Indeed, the well-known Nugent instances of this problem with up to 30 departments were solved to proven optimality in [3] using vast amounts of computational power and important improvements in mathematical programming algorithms.

In this paper, we consider a related special case of facility layout, namely the placement of facilities of given, and possibly different, dimensions on a straight line. This problem is known in the literature both as the linear single-row facility layout problem, see for example [14], and as the one-dimensional space allocation problem (ODSAP), see [29]. We shall refer to it as the ODSAP. An instance of the ODSAP consists of $n$ one-dimensional facilities $\{r_1, \ldots, r_n\}$ for which we are given their positive lengths $\ell_1, \ldots, \ell_n$ as well as their pairwise connectivities, $c_{ij}$. We are interested in finding an arrangement of the facilities next to each other along a line so as to minimize the total weighted sum of the center-to-center distances between all pairs of facilities. Therefore, like the aforementioned QAP, the ODSAP is an optimization problem over all possible permutations of the given facilities. In the special case where all the facilities have the same length, the ODSAP becomes a special case of the QAP [1,22]. Several applications of the ODSAP have been identified in the literature. These include the arrangement of books on a shelf, the layout of warehouses, and the layout of machines on a factory floor [14,26]. Furthermore, the ODSAP is closely related to the linear ordering problem, which also has a number of practical applications. A summary of these can be found in [8], along with references to the relevant literature. It is worth pointing out that the model we introduce in Section 2 captures the same combinatorial structure as the linear ordering polytope studied in [9,27], except that our model is based on semidefinite programming (SDP) rather than linear programming.

Let $\pi = (\pi_1, \ldots, \pi_n)$ denote a permutation of the indices $[n] := \{1, 2, \ldots, n\}$ of the facilities, so that the leftmost facility is $r_{\pi_1}$, the facility to the right of it is $r_{\pi_2}$, and so on, with $r_{\pi_n}$ being the last facility in the arrangement. Given a permutation $\pi$ and two distinct facilities $r_i$ and $r_j$, the center-to-center distance between $r_i$ and $r_j$ with respect to this permutation is

$$\tfrac{1}{2}\ell_i + D_\pi(i, j) + \tfrac{1}{2}\ell_j,$$

where $D_\pi(i, j)$ denotes the sum of the lengths of the facilities between $r_i$ and $r_j$ in the linear arrangement defined by $\pi$. To solve the ODSAP, we seek a permutation of the facilities which minimizes the weighted sum of the distances between all pairs of facilities. In mathematical terms, we wish to

$$\min_{\pi\in\Pi} \sum_{i<j} c_{ij} \left[\frac{1}{2}\,\ell_i + D_\pi(i, j) + \frac{1}{2}\,\ell_j\right],$$

where $\Pi$ denotes the set of all permutations $\pi$ of $[n]$.

The ODSAP was first studied by Simmons [29] who observed that it is possible to simplify the objective by eliminating the half-facility lengths. Indeed, we can rewrite the objective function as

$$\min_{\pi\in\Pi} \sum_{i<j} c_{ij} D_\pi(i, j) + \sum_{i<j} \frac{1}{2}\, c_{ij}(\ell_i + \ell_j),$$

where the second summation is a constant independent of $\pi$. We will thus focus our attention on optimizing $\sum_{i<j} c_{ij} D_\pi(i, j)$ over all permutations $\pi$.

Another observation concerns the symmetry of the arrangements. It is clear that

$$D_\pi(i, j) = D_{\pi'}(i, j),$$

where $\pi'$ denotes the permutation symmetric to $\pi$, defined by

$$\pi'(i) = \pi(n + 1 - i) \quad \text{for } i = 1, \ldots, n.$$

It follows that for the ODSAP, we can exchange the left and right ends of the layout and obtain the same objective value. Hence, it possible to simplify the problem by considering only the permutations for which, say, $r_1$ is on the left half of the arrangement. This type of symmetry-breaking strategy is important for reducing the computational requirements of most algorithms, including