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On new characterizations of the Owen value

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ABSTRACT

The aim of this paper is to provide new characterizations for the Owen value that generalize different characterizations of the Shapley value. To obtain these characterizations we consider axioms of efficiency, balanced contributions, symmetry, strong monotonicity, additivity and null player in the context of cooperative games with coalition structure.

1. Introduction

The Shapley value is one of the most famous solutions in cooperative games. This solution was defined and first characterized by means of axioms of carrier, additivity and anonymity in Shapley [14]. A close characterization of this solution with efficiency, additivity, symmetry and null player can be found in Shubik [15]. Although there are many other interesting characterizations of the Shapley value, two characterizations that also deserve special attention in this paper are the ones by Young [18] by means of efficiency, symmetry and strong monotonicity and Myerson [12] with the axioms of efficiency and balanced contributions.

In some cases, the players are grouped forming a coalition structure, which is just a partition of the set of players. With the aim to study the cooperation of the players in these situations, Aumann and Drèze [4] introduced the class of cooperative games with coalition structure. They also defined a solution for this class of games which extends the Shapley value. According to this solution, the payoff of a player is given by the Shapley value of this player in the game restricted to the union he belongs to. But, although this solution still preserves interesting axioms, it does not satisfy efficiency.

Later on, Owen [13] defined an efficient solution for cooperative games with coalition structure that also extends the Shapley value. This efficient solution has been successfully applied in the Operations Research field. Some of these applications can be found in Carreras and Owen [7], where the process of coalition formation in the Catalonian Parliament is considered, in Vázquez-Brage et al. [16], by using the Owen value to determine the aircraft landing fees, or in Alonso-Meijide and Bowles [1], where the distribution of power in the International Monetary Fund is studied.

Since the Owen value is a very well-known coalitional value, there are many characterizations of this value in the literature. The first one was made by Owen. The axioms involved in the characterization are efficiency, additivity, coalitional symmetry, intracoalitional symmetry and null player. Other interesting characterizations are the one provided by Calvo et al. [6] to characterize the level structure value, and in particular the Owen value, according to efficiency and the principle of balanced contributions applied in the different levels or the characterization that appears in Khmelnitskaya and Yanovskaya [10], where the Owen value is characterized by means of efficiency, strong monotonicity, coalitional symmetry and intracoalitional symmetry.

In this paper new characterizations for the Owen value are obtained. All the characterizations make use of axioms that also follow the spirit of the axioms used to characterize the Shapley value in [14,18,12]. In particular, the axiom of intracoalitional balanced contributions, introduced in [6], is a key element. It says that, given two players in the same union, the amounts that both players gain or lose when the other leaves the game should be equal. Other papers where this axiom has been used to characterize the Owen value are Gómez-Rúa and Vidal-Puga [8], comparing the Owen value with other coalitional values, and Lorenzo–Freire [11], where the Owen value is compared with the Banzhaf–Owen value. In Vázquez-Brage et al. [16] and Alonso-Meijide et al. [2] the principle of balanced contributions is also applied to get characterizations of the Owen value, but with a

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slightly different meaning since in this case the players leave the union but not the game.

The paper is organized as follows. In Section 2, the concepts used along the paper are introduced and, in Section 3, the results of the paper are collected.

2. Notation and definitions

2.1. TU-games

A *cooperative game with transferable utility* (or *TU-game*) is a pair (N, v) defined by a finite set of players $N \subset \mathbb{N}$ (usually, $N = \{1, 2, \ldots, n\}$) and a function $v : 2^N \to \mathbb{R}$, that assigns to each coalition $S \subseteq N$ a real number $v(S)$, called the worth of S, and such that $v(\emptyset) = 0$. For any coalition $S \subseteq N$, we assume that $s = |S|$. $\mathcal{G}_N$ will denote the family of all TU-games on a given N and $\mathcal{G}$ the family of all TU-games.

Given $S \subseteq N$, we denote the restriction of a TU-game $(N, v) \in \mathcal{G}_N$ to S as (S, v).

Given two TU-games $(N, v), (N, v') \in \mathcal{G}_N$, the TU-game $(N, v + v')$ is defined by $(v + v')(S) = v(S) + v'(S)$ for all $S \subseteq N$.

Given $\emptyset \neq T \subseteq N$, the *unanimity game* $\left(N, u^T\right)$ is a TU-game such that $u^T(S) = 1$ if $T \subseteq S$ and $u^T(S) = 0$ otherwise. Since the family $\left\{(N, u^T)\right\}_{T \in 2^N \setminus \emptyset}$ is a basis for $\mathcal{G}_N$, for any TU-game (N, v), there are unique coefficients $\{c_T\}_{T \in 2^N \setminus \emptyset}$ such that $v = \sum_{T \in 2^N \setminus \emptyset} c_T u^T$.

A player $i \in N$ is a *null player* in the TU-game (N, v) if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \setminus \{i\}$. The set of null players in (N, v) will be denoted by $\mathcal{NP}(N, v)$.

Two players $i, j \in N$ are *symmetric* in the TU-game (N, v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$.

A *value* is a map f that assigns to every TU-game $(N, v) \in \mathcal{G}$ a vector $f(N, v) = (f_i(N, v))_{i \in N}$, where each component $f_i(N, v)$ represents the payoff of player i when he participates in the game.

The *Shapley value* (Shapley [14]) is the value defined for all $(N, v) \in \mathcal{G}$ and all $i \in N$ by $Sh_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} [v(S \cup \{i\}) - v(S)]$.

Given an unanimity game $\left(N, u^T\right)$ for $\emptyset \neq T \subseteq N$, the expression of the Shapley value in this particular TU-game is given by $Sh_i\left(N, u^T\right) = \frac{1}{t}$ if $i \in T$ and $Sh_i\left(N, u^T\right) = 0$ otherwise.

2.2. TU-games with coalition structure

Let us consider a finite set of players N. A *coalition structure* over N is a partition of N, i.e., $C = \{C_1, \ldots, C_m\}$ is a coalition structure over N if it satisfies that $\bigcup_{h \in M} C_h = N$, where $M = \{1, \ldots, m\}$, and $C_h \cap C_r = \emptyset$ when $h \neq r$.

A *cooperative game with coalition structure* (or *TU-game with coalition structure*) is a triple (N, v, C) where (N, v) is a TU-game and C is a coalition structure over N. The set of all TU-games with coalition structure will be denoted by $\mathcal{CG}$, and by $\mathcal{CG}_N$ the subset where N is the player set.

Given $S \subseteq N$, such that $S = \bigcup_{h \in M} S_h$, with $\emptyset \neq S_h \subseteq C_h$ for all $h \in M$, we will denote the restriction of $(N, v, C) \in \mathcal{CG}_N$ to S as the TU-game with coalition structure (S, v, C_S), where $C_S = \{S_1, \ldots, S_m\}$.

If $(N, v, C) \in \mathcal{CG}$ and $C = \{C_1, \ldots, C_m\}$, the *quotient game* $\left(M, v^C\right)$ is the TU-game played by the unions where the player set is given by $M = \{1, \ldots, m\}$. It is defined as $v^C(R) = v\left(\bigcup_{r \in R} C_r\right)$ for all $R \subseteq M$.

Let us consider $(N, v, C) \in \mathcal{CG}$. A union $C_h \in C$ is a *null union* in (N, v, C) if h is a null player in $\left(M, v^C\right)$. Two unions, $C_h, C_r \in C$, are *symmetric unions* in (N, v, C) if h and r are symmetric players in $\left(M, v^C\right)$.

A *coalitional value* is a map g that assigns to every TU-game with coalition structure (N, v, C) a vector $g(N, v, C) = (g_i(N, v, C))_{i \in N}$.

The *Owen value* (Owen [13]) is the coalitional value defined for all $(N, v, C) \in \mathcal{CG}$ and all $i \in C_h$, with $C_h \in C$, by $Ow_i(N, v, C) = \sum_{R \subseteq M \setminus \{h\}} \sum_{S \subseteq C_h \setminus \{i\}} \frac{r!(m-r-1)!}{m!} \frac{s!(c_h-s-1)!}{c_h!} [v(\bigcup_{r \in R} C_r \cup S \cup \{i\}) - v(\bigcup_{r \in R} C_r \cup S)]$.

3. The characterizations

Below, we introduce the axioms used to characterize the Owen value.

- *Additivity (ADD).* For all $(N, v, C), (N, v', C) \in \mathcal{CG}$, $g(N, v + v', C) = g(N, v, C) + g(N, v', C)$.
- *Coalitional marginality (CMA).* For all $(N, v, C), (N, v', C) \in \mathcal{CG}$, if $v(S \cup C_h) - v(S) = v'(S \cup C_h) - v'(S)$ for all $S \subseteq N \setminus C_h$ then $\sum_{i \in C_h} g_i(N, v, C) = \sum_{i \in C_h} g_i(N, v', C)$.
- *Coalitional strong monotonicity (CSM).* For all $(N, v, C), (N, v', C) \in \mathcal{CG}$, if $v(S \cup C_h) - v(S) \geq v'(S \cup C_h) - v'(S)$ for all $S \subseteq N \setminus C_h$ then $\sum_{i \in C_h} g_i(N, v, C) \geq \sum_{i \in C_h} g_i(N, v', C)$.
- *Coalitional symmetry (CSY).* For all $(N, v, C) \in \mathcal{CG}$ and for all symmetric coalitions $C_h, C_r \in C$, $\sum_{i \in C_h} g_i(N, v, C) = \sum_{i \in C_r} g_i(N, v, C)$.
- *Efficiency (EFF).* For all $(N, v, C) \in \mathcal{CG}$, $\sum_{i \in N} g_i(N, v, C) = v(N)$.
- *Intracoalitional balanced contributions (IBC).* For all $(N, v, C) \in \mathcal{CG}$ and all $i, j \in C_h \in C$, $i \neq j$, $g_i(N, v, C) - g_i(N \setminus \{j\}, v, C_{N \setminus \{j\}}) = g_j(N, v, C) - g_j(N \setminus \{i\}, v, C_{N \setminus \{i\}})$.
- *Individual marginality (IMA).* For all $(N, v, C), (N, v', C) \in \mathcal{CG}$, if $v(S \cup \{i\}) - v(S) = v'(S \cup \{i\}) - v'(S)$ for all $S \subseteq N \setminus \{i\}$ then $g_i(N, v, C) = g_i(N, v', C)$.
- *Individual strong monotonicity (ISM).* For all $(N, v, C), (N, v', C) \in \mathcal{CG}$, if $v(S \cup \{i\}) - v(S) \geq v'(S \cup \{i\}) - v'(S)$ for all $S \subseteq N \setminus \{i\}$ then $g_i(N, v, C) \geq g_i(N, v', C)$.
- *Null player (NP).* For all $(N, v, C) \in \mathcal{CG}$ and for all $i \in N$, if i is a null player then $g_i(N, v, C) = 0$.
- *Null union (NU).* For all $(N, v, C) \in \mathcal{CG}$ and for all $C_h \in P$, if C_h is a null union then $\sum_{i \in C_h} g_i(N, v, C) = 0$.

The axioms of ADD, EFF, CSY and NP are quite standard in the literature and they have been used in several characterizations of different coalitional values.

The null union axiom is used in Calvo and Gutiérrez [5] to characterize the two-step Shapley value. They also show that there is no relation between NP and NU.

CMA and IMA are adaptations of the axiom of marginality defined by Young [18] to the context of cooperative games with coalition structure, whereas CSM and ISM follow the spirit of strong monotonicity.

IBC was defined in [6] and states that, given two players in the same union, the amounts that both players gain or lose when the other leaves the game should be equal. In Lorenzo-Freire [11], a new expression based on the Shapley value is obtained for all the coalitional values satisfying intracoalitional balanced contributions.

Proposition 3.1 (*Lorenzo-Freire [11]*)**.** *A coalitional value g satisfies IBC if and only if, for all $(N, v, C) \in \mathcal{CG}$ and all $i \in C_h$ with $C_h \in C$,*

$$g_i(N, v, C) = Sh_i\left(C_h, v^{g,C}\right),$$

where $v^{g,C}(S) = \sum_{i \in S} g_i\left((N \setminus C_h) \cup S, v, C_{(N \setminus C_h) \cup S}\right)$ *for all* $S \subseteq C_h$.

In Young [18], the Shapley value is characterized by means of efficiency, symmetry and marginality. In the next proposition, EFF and CSY are combined with the two adaptations of marginality to the context of cooperative games with coalition structure. In both cases, we obtain that the total payoff of the players in each union is given by the Shapley value of the union in the quotient game.