Operations Research Letters 44 (2016) 498–502

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Shelf life of candidates in the generalized secretary problem☆

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ARTICLE INFO

Article history:
Received 11 May 2015
Received in revised form
12 May 2016
Accepted 12 May 2016
Available online 20 May 2016

Keywords:
Optimal stopping
Relative ranks
Best-choice problem
Duration problem
Dynamic programming

ABSTRACT

The study presents a version of the secretary problem called the duration problem in which the objective is to maximize the time of possession of the relatively best or the second best objects. It is shown that in this duration problem there are threshold numbers such that the optimal strategy is determined by them. When the number of objects tends to infinity the thresholds values are $\lfloor 0.120381N \rfloor$ and $\lfloor 0.417188N \rfloor$, respectively, and the asymptotic mean time of shelf life is $0.403827N$.

1. Introduction and summary

The concept of duration which appears in the BCP and its generalization has two different meanings. The *first meaning* refers to the time in which a certain characteristic is immutable. Let us explain it as follows. The leaders of sports discipline change over time. We say that Player A, who achieved the best results in the competition and by the way broke the record of the discipline, meaning not only won the race, but also became the owner of the record. This result will be subsequently corrected at the official competitions. We note this player for the time in which he was a master. It is his perseverance in the position as a master, his being and the duration at the position of a record holder which are considered. If the day of his record is k and the new record moment is $T_k \geq k$, then his duration is $T_k - k$. The moment T_k is the moment of maturity of the contract or the moment of extinguishing the rights to the title. This idea of duration problem for the classical no-information secretary problem has been investigated for the first time by Ferguson, Hardwich and Tamaki [4]. It is a sequential selection problem which is a variation of the classical secretary problem (CSP), treated for example, by Gilbert and Mosteller [6]. The aim in CSP is to examine items ranked from 1 to N by random selection without replacement, one at a time, and in order to win is to stop at any item whose overall (absolute) rank belongs to the given set of ranks (in the basic version this set contains the rank 1 only), given only the relative ranks of the items drawn so far. Since the appearance of Gardner's articles [5] the secretary problem has been extended and generalized in many different directions. Excellent reviews of the development of this colorful problem and its extensions have been given by Samuels [20] and Ferguson [3]. The basic form of the duration problem for CSP is to select the relatively best object, receiving a pay-off of one as we do so and an additional one for each new observation as long as the selected object remains relatively best.

The *second meaning* refers to the time in which we sought a candidate, and so the time to observe and identify the leader. If this is the criterion of the quality of the elected candidate, there is no doubt that not every applicant will be accepted. Even if we choose it cannot meet the established criteria. The time from the start of the search until the election of a candidate being worth it, is also named duration—this time the duration of the search. It was Quine and Law [19], who considered such a version of the secretary problem. The candidates ranked from 1 to N are randomly selected without replacement, one at a time. The success is success if the item is chosen from among the s best, given only the relative ranks so far. The optimal stopping time $\tau^\star$ for this problem is the s-tuple threshold Markov moment. This stopping time $\tau^\star$ is called duration by Yeo and Yeo [25].

The article considers the issue, which is related to the first of the mentioned ways of comprehending the duration time. Though

☆ Announcement of the topic is on arXiv.

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http://dx.doi.org/10.1016/j.orl.2016.05.002
0167-6377/

Ferguson, Hardwich and Tamaki [4] examined the various duration models extensively, they confined themselves to the study of the shelf life of the relatively best items. The list of no-information case problems considered there is presented in Section 2. In this paper, we attempt to extend the problems to choose and keep one of two best items. To this end the no-information duration models of [4] will be presented as the optimal stopping problems of an embedded Markov chain similarly as in the papers related to the generalization of the BCP (see [10,26,23,22]). For simplicity we refer to the relatively best or the second best object as a candidate (see [6,11,24],). The focus is on the case where each time we receive a unit pay-off as long as either of the chosen objects remains a **candidate**. Obviously only candidates can be chosen, the objective being to maximize expected pay-off. This problem can be viewed from another perspective as follows. Let us observe at moment i the relatively second candidate and let us denote by $T(i)$ the time of appearance of the first new candidate after moment i (*i.e.* the relatively best or the second best item) if there is one, and $N+1$ if there is none. If we observe at i the relatively best item then $T(i)$ is the moment when a new item appears which changes the relative rank of ith item to the no candidate rank (various meanings of a *candidate* or restriction on the candidate can be modeled by $\xi_i = \rho$ (relative ranks of applicants presented so far). The time $T(i)-i$ is called duration of the candidate selected at time i. The objective is to find a stopping time $\tau^\star$ such that

$$v_N = E\left[\frac{T(\tau^*)-\tau^*}{N}\xi_{\tau^\star}\right] = \sup_{\tau\in\mathfrak{M}^N} E\left[\frac{T(\tau)-\tau}{N}\xi_\tau\right], \quad (1)$$

where τ denotes the stopping time and $\mathfrak{M}^N$ denotes the set of all Markov times.

This problem will be discussed in Section 2. The optimal strategy will be derived in Section 2.3. It can be shown, based on the suggestion from [2] and the results by [23,22], that there exists an optimal threshold stopping time such that it immediately selects the best candidate if it appears after or on time $k_1^\star$ and it immediately selects the second best candidate if it appears after time $k_2^\star > k_1^\star$. In Section 3 we investigate the asymptotics as $n \to \infty$. $k_1^\star/n$ proves to converge to $a \cong 0.120381$ and $k_2^\star/n$ to $b \cong 0.417188$. The asymptotic mean time of shelf life of the relatively best or the second best object is $0.403827N$. Some details of the calculations and part of the proofs have been placed in the supplementary materials.

2. Markov model for the shelf life of the best and the second best

The models which are considered in this study are so called no information models where the decision to select an object is based only on the relative ranks of the objects observed so far. Let $\mathbb{S}=$ be the set of ranks of items $\{x_1,x_2,\ldots,x_N\}$ and $\{X_1,X_2,\ldots,X_N\}$ their permutation. We assume that all permutations are equally likely. If X_k is the rank of kth candidate we define

$$Y_{kj} = \#\{1\le i\le j : X_i \le X_k\}$$

the running rank of kth object at moment $j\ge k$. The random variable $Y_k=Y_{kk}$ is called *relative rank* of kth candidate with respect to the items investigated to the moment k. Let $A\subset\mathbb{S}$. Next the appearance of moment $S_A(i)=\inf\{k>i : Y_k\in A\}$ of the next candidate after i and the maturity $T_r(i) = \inf\{k\ge i : \mathbb{P}\{X_k \in A|Y_{ik}\}=0\}$ of the candidate with the relative rank r at moment i are defined.

Remark 1. If $A=\{1,2,\ldots,s\}$ and $S_A(i)=\inf\{k>i : Y_k\in A\}$ then $T_s(i)=S_A(i)$ and for any $r\in A$ $T_r(i)=\inf\{k\ge i : Y_{ik}\notin A\}$. For $r<s$ the duration of the candidate with rank r at moment i is dependent on items appeared between i and k. Changing the rank of a candidate when new candidate approaches, does not always mean that ceased to be a candidate.

We observe sequentially the permutation of items from the set $\mathbb{S}$. The mathematical model of such an experiment is the probability space $(\Omega,\mathcal{F},\mathbf{P})$. The elementary events are permutations of the elements from $\mathbb{S}$ and the probability measure $\mathbf{P}$ is the uniform distribution on Ω. The observation of random variables Y_k, $k=1,2,\ldots,N$, generates the sequence of σ-fields $\mathcal{F}_k=\sigma\{Y_1,Y_2,\ldots,Y_k\}$, $k=1,2,\ldots,N$. The random variables Y_k are independent and $\mathbf{P}\{Y_k=i\}=\frac{1}{k}$.

Denote by $\mathfrak{M}^N$ the set of all Markov moments τ with respect to σ-fields $\{\mathcal{F}_k\}_{k=1}^N$. The decision maker observes the stream of relative ranks. When $Y_i\in A=\{1,2,\ldots,s\}$ it is the potential candidate for the absolutely rth item, $r\in A$. Sometimes it is enough to keep such a candidate for a period of time to get profit which is proportional to the shelf life of a candidate (the second kind of duration of the candidate). The random variable $T(i)$ is defined as the moment when the keeping candidate stops to be the candidate (the maturity of the candidate). Considering the possibility of recall forces us to think about rejected candidates. We define $\delta_r(i) = \sup_{1\le j\le i}\{Y_j=r\}$ as the actual position of the relative r at moment i. $\delta_r(i)$ is the random variable measurable with respect to $\mathcal{F}_i$. The recall option means the possibility of returning to the last candidate who has the relative rank $r^\star = \arg\max_{\{s\in A\}}\delta_s(i)$ or to the candidate with the given rank, e.g $r=1$. The present history at i and $Y_{k,s}$ for $k\in\{i,\ldots,N\}$, $s=i,i+1,\ldots,N$ allows to define the maturity $T(i)$ for various models. In the next part examples of the various definitions of maturity will be shown, and therefore different definitions of duration will be presented.

2.1. Classical no-information BC (Best Choice) duration problem

It is not difficult to formalize the duration problem for BC as without recall as with recall and also when the additional requirement concerning the absolute rank of the selected object is added. As to present the problem of duration time for BCP considered in [1] we assume that $A=\{1\}$, $\zeta_n(\omega)=\mathbb{I}_{\{Y_n\in A\}}(\omega)$ and $\zeta_n^\star(\omega)=\mathbb{I}_{\{X_n\in A\}}(\omega)$.

2.1.1. Finite horizon duration problem of BCP (Best Choice Problem) without recall ([4, Sec. 2.2])

Let $T(i)=\zeta_i T_1(i)$ and $\xi_i=1$. The aim is to find $\tau^\star\in\mathfrak{M}^N$ such that:

$$\mathbf{E}\left[\frac{T(\tau^\star)-\tau^\star}{N}\xi_{\tau^\star}\right] = \sup_{\tau\mathfrak{M}^N}\mathbf{E}\left[\frac{(T(\tau)-\tau)}{N}\xi_\tau\right]. \quad (2)$$

It is the first setting of the problem. In [4] the authors observed that the pay-offs in the problem for the threshold rules are exactly same as the pay-offs for the threshold rule for the BC secretary problem with an unknown, random number of options having the uniform distribution on $\mathbb{S}$ (see a general method of Samuels [20] showing the relation of the random horizon problems to the problems with cost). The single threshold $r_N^\star$ strategy is optimal having the asymptotic $\lim_{N\to\infty}\frac{r_N^\star}{N}\cong e^{-2}$ and the problem value $2e^{-2}$.

2.1.2. Finite horizon duration problem of BCP with recall ([4, Sec. 2.2])

Let $T(i)=T_1(i)$ and $\xi_i=1$. The aim is to find $\tau^\star\in\mathfrak{M}^N$ in the problem of (2) with this new definition of the maturity moment. This second setting of the problem has the solution which has a simple relation with the solution of BCP. Namely, if $k_N^\star$ is the optimal threshold for BCP then the optimal threshold for the duration problem of BCP with recall is $K_N = r_N^\star - 1$ for $N\ge 2$. It is also the optimal rule for the BCP with an unknown, random number of options having the uniform distribution on $\mathbb{S}$ and possible recall.