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Characterization sets for the nucleolus in balanced games

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ABSTRACT

We provide a new modus operandi for the computation of the nucleolus in cooperative games with transferable utility. Using the concept of dual game we extend the theory of characterization sets. Dually essential and – if the game is monotonic – dually saturated coalitions determine both the core and the nucleolus whenever the core is non-empty. We show how these two sets are related to the existing characterization sets. In particular we prove that if the grand coalition is vital then the intersection of essential and dually essential coalitions forms a characterization set itself.

1. Introduction

The nucleolus, introduced by Schmeidler [29], is one of the most frequently applied cooperative game theoretical solution concept. Much like the Shapley-value it suffers from computational difficulties. Determining the nucleolus is $\mathcal{NP}$-hard for various classes of games such as minimum cost spanning tree games [7], voting games [6] and flow and linear production games [5]. Recently Greco et al. [13] provided a non-trivial upper bound for its complexity. On the other hand there are known polynomial time algorithms for computing the nucleolus of important families of cooperative games, like standard tree [21], assignment [31], matching [16] and bankruptcy games [2]. In addition, Kuipers [18] and Arin and Inarra [1] developed methods to compute the nucleolus for convex games.

In contrast to the Shapley-value, there is no formula for the nucleolus. One way to compute it is to use linear programming. The sequential LP approach of Maschler et al. [20] was the first computationally tractable one. Since then there have been many attempts to improve the computation process see e.g. [28,8,25,26]. Although all the proposed LPs consist of exponentially many inequalities they can be solved efficiently if one knows which constraints are redundant. Huberman [15], Granot et al. [12] and Reijnierse and Potters [27] provided methods to identify coalitions that correspond to non-redundant constraints.

This idea was already suggested by Megiddo [23], but the formal theoretical framework was developed by Granot et al. [12]. They introduced the concept of characterization sets, namely a collection of coalitions that itself determines the nucleolus, and proved that if the size of the characterization set is polynomially bounded in the number of players, then the nucleolus of the game can be computed in strongly polynomial time. A collection that characterizes the nucleolus in a game need not characterize it in another game. Thus we are interested in a property of the coalitional function that describes a characterization set in every game in a sufficiently large class of games. Huberman [15] was the first to show that such a property exists in the very important class of balanced games. He introduced the concept of essential coalitions which are coalitions that have no weakly majorizing partition, and proved that if the core is non-empty, the family of essential coalitions itself determines the nucleolus. Granot et al. [12] provided another collection that characterize the nucleolus in cost games with non-empty core.

We introduce two new characterization sets: dually essential and dually saturated coalitions. We show that both sets (in themselves) determine the core, and if the core is non-empty they determine the nucleolus as well. We conclude by analyzing the relationship of the four known characterizing properties.

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2. Game-theoretic framework

A *cooperative game with transferable utility* is an ordered pair (N, v) consisting of the player set $N = \{1, 2, \ldots, n\}$ and a characteristic function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. The value $v(S)$ represents the worth of coalition S. Let $\mathcal{P} = 2^N \setminus \{\emptyset, N\}$ denote the family of the non-trivial coalitions. A cooperative game (N, v) is called *monotonic* if $S \subseteq T \subseteq N \Rightarrow v(S) \le v(T)$. A general assumption in cooperative games is that the grand coalition (N) forms. The question is then how to distribute $v(N)$ among the players in some fair way.

An outcome for a cooperative game $\Gamma = (N, v)$ is a vector $x \in \mathbb{R}^N$ that represents the payoff of each player. For convenience, we introduce the notation $x(S) = \sum_{i \in S} x_i$ for any $S \subseteq N$, and instead of $x(\{i\})$ we simply write $x(i)$. A solution is called *efficient* if $x(N) = v(N)$ and *individually rational* if $x(i) \ge v(i)$ for all $i \in N$. Efficient solutions will also be called *allocations*. The imputation set $\mathbf{I}(\Gamma)$ of the game Γ consists of the efficient and individually rational solutions, formally, $\mathbf{I}(\Gamma) = \{x \in \mathbb{R}^N \mid x(N) = v(N),\ x(i) \ge v(i) \text{ for all } i \in N\}$. Given a payoff vector $x \in \mathbb{R}^N$, we define the *satisfaction* of coalition S in game Γ as

$$sat_\Gamma(S, x) := x(S) - v(S).$$

The satisfaction of the grand coalition is clearly zero at any allocation. The core $\mathbf{C}(\Gamma)$ of cooperative game Γ is the set of allocations where all the satisfaction values are non-negative. Formally,

$$\mathbf{C}(\Gamma) = \{x \in \mathbb{R}^N \mid sat_\Gamma(N, x) = 0,\ sat_\Gamma(S, x) \ge 0 \text{ for all } S \in \mathcal{P}\}.$$

A game is called *balanced* if its core is non-empty. In this paper we will consider only balanced games. Notice that core vectors of monotonic games are non-negative. Indeed, $x(i) \ge v(i) \ge v(\emptyset) = 0$ for all $i \in N$.

We say that a vector $x \in \mathbb{R}^m$ *lexicographically precedes* $y \in \mathbb{R}^m$ (denoted by $x \preceq y$) if either $x = y$ or there exists a number $1 \le j \le m$ such that $x_i = y_i$ if $i < j$ and $x_j < y_j$. Let $\Gamma = (N, v)$ be a game and let $\theta^{\mathcal{P}}(x) \in \mathbb{R}^{2^n - 2}$ be the vector that contains the $2^n - 2$ satisfaction values $sat_\Gamma(S, x)$, $S \in \mathcal{P}$, in a non-decreasing order.

Definition 1. The *nucleolus* of a cooperative game Γ is the subset of the payoff vectors $x \in \mathbf{I}(\Gamma)$ that lexicographically maximize $\theta^{\mathcal{P}}(x)$. Formally,

$$\mathbf{N}(\Gamma) = \{x \in \mathbf{I}(\Gamma) \mid \theta^{\mathcal{P}}(y) \preceq \theta^{\mathcal{P}}(x)\ \forall y \in \mathbf{I}(\Gamma)\}.$$

Schmeidler [29] proved that $\mathbf{N}(\Gamma)$ consists of a single point, and that it is a continuous function of the characteristic function. Although formally a set, we will consider $N(\Gamma)$ as an allocation vector. It is straightforward that $\mathbf{N}(\Gamma) \in \mathbf{C}(\Gamma)$ whenever $\mathbf{C}(\Gamma)$ is non-empty.

Kohlberg [17] offered a method to verify whether an allocation coincides with the nucleolus. Let $e_S \in \{0, 1\}^N$ denote the membership vector of coalition S, that is, $(e_S)_i = 1$ if $i \in S$ and $(e_S)_i = 0$ otherwise. A collection of coalitions $\mathcal{B}_S \subseteq \mathcal{P}$ is said to be *S-balanced* if there exist positive weights λ_T, $T \in \mathcal{B}_S$, such that $\sum_{T \in \mathcal{B}_S} \lambda_T e_T = e_S$. An N-balanced collection is simply called *balanced*.

Theorem 2 (*Kohlberg [17]*)**.** *Let $\Gamma = (N, v)$ be a game. Then an allocation x is the nucleolus if and only if for all $y \in \mathbb{R}$ the collection $\{S \in \mathcal{P} \mid sat_\Gamma(S, x) \le y\}$ is balanced or empty.*

3. Characterization sets

In the following we will use the formalism of Granot et al. [12].

Definition 3. Let $\Gamma^{\mathcal{F}} = (N, \mathcal{F}, v)$ be a cooperative game with coalition formation restrictions, where $\emptyset \ne \mathcal{F} \subseteq \mathcal{P}$ consists of all coalitions deemed permissible besides the grand coalition N. Let $\theta^{\mathcal{F}}(x) \in \mathbb{R}^{|\mathcal{F}|}$ be the restricted vector that contains the satisfaction values $sat_\Gamma(S, x)$, $S \in \mathcal{F}$ in a non-decreasing order. Furthermore, let $\mathbf{N}(\Gamma^{\mathcal{F}})$ be defined as the set of allocations that lexicographically maximize $\theta^{\mathcal{F}}(x)$. Then $\mathcal{F}$ is called a characterization set for the nucleolus of the game $\Gamma = (N, v)$, if $\mathbf{N}(\Gamma^{\mathcal{F}}) = \mathbf{N}(\Gamma)$.

Note the different roles of N and the coalitions in $\mathcal{F}$ in the above optimization. The satisfaction of the grand coalition is required to be constantly zero (defining the feasible set), whereas the satisfactions of the smaller permissible coalitions form the objective function. Notice that $\mathbf{N}(\Gamma^{\mathcal{F}})$ is a singleton if and only if $rank(\{e_S : S \in \{N\} \cup \mathcal{F}\}) = n$. Most relevant to our setting is the following theorem of Granot et al. [12].

Theorem 4. *Let Γ be a cooperative game with a non-empty core. The non-empty collection $\mathcal{F} \subseteq \mathcal{P}$ is a characterization set for the nucleolus of Γ if for every $S \in \mathcal{P} \setminus \mathcal{F}$ there exists a non-empty subcollection $\mathcal{F}_S$ of $\mathcal{F}$, such that*

i. *$sat_\Gamma(T, x) \le sat_\Gamma(S, x)$ for all $x \in \mathbf{C}(\Gamma)$, whenever $T \in \mathcal{F}_S$,*
ii. *e_S can be expressed as a linear combination of the vectors in $\{e_T : T \in \mathcal{F}_S \cup \{N\}\}$.*

Observe that the above conditions are sufficient but not at all necessary. Take for example the (superadditive and balanced) profit game with four players $N = \{1, 2, 3, 4\}$ and the following coalitional function: $v(i) = 0$, $v(i, j) = 1$, $v(i, j, k) = 3$ for any $i, j, k \in N$ and $v(N) = 4$. Then the 2-player coalitions and the grand coalition are sufficient to determine the nucleolus, which is given by $z(i) = 1$ for all $i \in N$. However, the 3-player coalitions have smaller satisfaction values at z, thus the first condition of Theorem 4 is violated. Notice that in this game the 3-player coalitions and the grand coalition are also sufficient to determine the nucleolus.

In general neither the 2-player nor the 3-player coalitions (and the grand coalition) characterize the nucleolus. The fact that in this example they did was due to the particular choice (most importantly the symmetry) of the coalitional function. We would like to identify properties of coalitions that characterize the nucleolus independently of the realization of the coalitional function.

The Kohlberg-criterion applied to games with coalition formation restrictions yields the following theorem.

Theorem 5 (*Maschler et al. [22]*)**.** *Let $\mathcal{F}$ be a characterization set and x be an imputation of the game Γ with $\mathbf{C}(\Gamma) \ne \emptyset$. Then $x = \mathbf{N}(\Gamma)$ if and only if for all $y \in \mathbb{R}$ the collection $\{S \in \mathcal{F} \mid sat_\Gamma(S, x) \le y\}$ is balanced or empty.*

A similar criterion appears in [14]. With the help of the Kohlberg-criterion the problem of finding the nucleolus is reduced to finding the right characterization set. The first characterization set for balanced games is due to Huberman [15].

Definition 6 (*Essential Coalitions*)**.** Let N be a set of players, (N, v) a cooperative game. Coalition $S \in \mathcal{P}$ is called *essential* in game $\Gamma = (N, v)$ if it cannot be partitioned as $S = S_1 \dot\cup \ldots \dot\cup S_k$ with $k \ge 2$ and $S_j \ne \emptyset$ for all $1 \le j \le k$ such that S is weakly majorized by $S_1, \ldots, S_k$, that is

$$v(S) \le v(S_1) + \cdots + v(S_k).$$

The set of essential coalitions is denoted by $\mathcal{E}(\Gamma)$. A coalition that is not essential is called *inessential*.

A similar criterion can be formulated for cost games, but there the inequality is reversed, i.e. $c(S) \ge c(S_1) + \cdots + c(S_k)$ must hold.